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ANALYSIS AND GEOMETRY ON MARKED CONFIGURATION SPACES

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Abstract

We carry out analysis and geometry on a marked configuration space Ω_X^M over a Riemannian manifold X with marks from a space M. We suppose that M is a homogeneous space M of a Lie group G. As a transformation group $\mathfrak{A}$ on Ω_X^M we take the "lifting" to Ω_X^M of the action on $X\times M$ of the semidirect product of the group $\mathrm{Diff}_0(X)$ of diffeomorphisms on X with compact support and the group G^X of smooth currents, i.e., all C^∞ mappings of X into G which are equal to the identity element outside of a compact set. The marked Poisson measure π_σ on Ω_X^M with Lévy measure σ on $X\times M$ is proven to be quasiinvariant under the action of $\mathfrak{A}$. Then, we derive a geometry on Ω_X^M by a natural "lifting" of the corresponding geometry on $X\times M$. In particular, we construct a gradient ∇^Ω and a divergence $\operatorname{div}^\Omega$. The associated volume elements, i.e., all probability measures μ on Ω_X^M with respect to which ∇^Ω and $\operatorname{div}^\Omega$ become dual operators on $L^2(\Omega_X^M;\mu)$, are identified as the mixed marked Poisson measures with mean measure equal to a multiple of σ. As a direct consequence of our results, we obtain marked Poisson space representations of the group $\mathfrak{A}$ and its Lie algebra $\mathfrak{a}$. We investigate also Dirichlet forms and Dirichlet operators connected with (mixed) marked Poisson measures.

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0 Introduction

In recent years, stochastic analysis and differential geometry on configuration spaces have been considerably developed in a series of papers [5–8], see also [37, 2, 3]. It has been shown, in particular, that the geometry of the configuration space Γ_X over a Riemannian manifold X can be constructed via a simple "lifting procedure" and is completely determined by the Riemannian structure of X. The mixed Poisson measures are then exhibited as the "volume elements" corresponding to the differential geometry introduced on Γ_X. Intrinsic Dirichlet forms and operators, their canonical processes, as well as Gibbs measures on configuration spaces, their characterization by integration by parts, and the corresponding stochastic dynamics are among the problems which have been treated in the above framework.

A starting point for this analysis, more exactly, for the definition of differentiation on the configuration space, was the representation of the group of diffeomorphisms $\mathrm{Diff}_0(X)$ on X with compact support that was constructed by G. A. Goldin et al. [18] and A. M. Vershik et al. [42] (see also [34, 38, 20]). The construction of this representation used, in turn, the fact, following from the Skorokhod theorem, that the Poisson measure is quasiinvariant with respect to the group $\mathrm{Diff}_0(X)$.

On the other hand, starting with the same work [42], many researchers consider representations also on marked (in particular, compound) Poisson spaces. In statistical mechanics of continuous systems, marked Poisson measures and their Gibbsian perturbations are used for the description of many concrete models, see e.g. [1]. Hence, it is natural to ask about geometry and analysis on marked Poisson spaces. The first work in this direction was the paper [26], in which, just as in the case of the usual Poisson measure, the action of the group $\mathrm{Diff}_0(X)$ was used for the definition of the differentiation. However, this group proved to be too small for reconstructing mixed marked Poisson measures as "volume elements," which means that $\mathrm{Diff}_0(X)$ is to be extended in a proper way, which we will describe in the present paper.

Let us recall that the configuration space Γ_X is defined as the space of all locally finite subsets (configurations) in X. Then, the marked configuration space Ω_X^M over X with marks from, generally speaking, a manifold M is defined as

$$\Omega_X^M := \{\, (\gamma, s) \mid \gamma \in \Gamma_X,\, s \in M^\gamma \,\},$$

where M^γ stands for the set of all maps $\gamma \ni x \mapsto s_x \in M$. Let $\widetilde{\sigma}$ be a Radon measure on $X \times M$ such that $\widetilde{\sigma}(K \times M) < \infty$ for each compact $K \subset X$ and $\widetilde{\sigma}$ is nonatomic in X, i.e., $\widetilde{\sigma}(\{x\} \times M) = 0$ for each $x \in X$. Then, one can define on Ω_X^M a marked Poisson measure $\pi_{\widetilde{\sigma}}$ with Lévy measure $\widetilde{\sigma}$.

Of course, one could consider $\pi_{\widetilde{\sigma}}$ as a usual Poisson measure on the configuration space $\Gamma_{X\times M}$ over the Cartesian product of the underlying manifold X and the space of marks M, and study the properties of this measure using the results of [2–5]. However, such an approach does not distinguish between the two different natures of X and M and the different roles that these play in physics. Thus, our aim is to introduce and study such transformations of the marked configuration space which do "feel" this difference and lead to an appropriate stochastic analysis and differential geometry.

In our previous paper [24], we were concerned with the model case $M = \mathbb{R}_+$, which corresponds, in fact, to the case of a compound Poisson measure. As has been promised in [24], we generalize in the present paper the results of [24] to the case where M is a homogeneous space of a Lie group G. This situation is natural from the physical point of view. For example, one can take $X = \mathbb{R}^3$ and M to be the unit sphere S^2 in $\mathbb{R}^3$, and consider any marked configuration $(\gamma, s) = \{(x, s_x)_{x\in\gamma}\} \in \Omega_X^M$ as a system of particles in $\mathbb{R}^3$ situated at the points x of γ and having spin s_x at $x \in \gamma$. One has then to take G as the rotation group, see e.g. [13].

Let G^X denote the group of smooth currents, i.e., all C^∞ mappings $X \ni x \mapsto \eta(x) \in G$ which are equal to the identity element of G outside of a compact set (depending on η). We define the group $\mathfrak{A}$ as the semidirect product of the groups $\mathrm{Diff}_0(X)$ and G^X: for $a_1 = (\psi_1, \eta_1)$ and $a_2 = (\psi_2, \eta_2)$, where $\psi_1, \psi_2 \in \mathrm{Diff}_0(X)$ and $\eta_1, \eta_2 \in G^X$, the multiplication of a_1 and a_2 is given by

$$a_1 a_2 = (\psi_1 \circ \psi_2, \eta_1(\eta_2 \circ \psi_1^{-1})).$$

The group $\mathfrak{A}$ acts in $X \times M$ as follows: for any $a = (\psi, \eta) \in \mathfrak{A}$

$$X \times M \ni (x, m) \mapsto a(x, m) = (\psi(x), \eta(\psi(x))m) \in X \times M,$$

where, for $g \in G$ and $m \in M$, gm denotes the action of g on m. Since each $\omega \in \Omega_X^M$ can be interpreted as a subset of $X \times M$, the action of $\mathfrak{A}$ can be lifted to an action in Ω_X^M. The marked Poisson measure $\pi_{\widetilde{\sigma}}$ is proven to be quasiinvariant under it. Thus, we can easily construct, in particular, a representation of $\mathfrak{A}$ in $L^2(\pi_{\widetilde{\sigma}})$. It should be stressed, however, that our representation of $\mathfrak{A}$ is reducible, because so is the regular representation of $\mathfrak{A}$ in $L^2(\widetilde{\sigma})$, see subsec. 3.5 in [24] for details.

Having introduced the action of the group $\mathfrak{A}$ on Ω_X^M, we proceed to derive analysis and geometry on Ω_X^M in a way parallel to the works [7, 24], dealing with the usual configuration space Γ_X and the marked configuration space $\Omega_X^{\mathbb{R}_+}$, respectively. In particular, we note that the Lie algebra $\mathfrak{a}$ of the group $\mathfrak{A}$ is given by $\mathfrak{a} = V_0(X) \times C_0^\infty(X; \mathfrak{g})$, where $V_0(X)$ is the algebra of C^∞ vector fields on X having compact support and $C_0^\infty(X; \mathfrak{g})$ is the algebra of C^∞ compactly supported functions from X into the Lie algebra $\mathfrak{g}$ of the group G. For each $(v, u) \in \mathfrak{a}$, we define the notion of a directional derivative of a function $F\colon \Omega_X^M \to \mathbb{R}$ along (v, u), which is denoted by $\nabla^\Omega_{(v,u)} F$. We obtain an explicit form of this derivative on the special set $\mathcal{F}C_{\mathrm{b}}^\infty\big(\mathfrak{D}, \Omega_X^M\big)$ of smooth cylinder functions on Ω_X^M, which, in turn, motivates our definition of a tangent bundle $T(\Omega_X^M)$ of Ω_X^M, and of a gradient $\nabla^\Omega F$. We note only that the tangent space $T_\omega(\Omega_X^M)$ to the marked configuration space Ω_X^M at a point $\omega = (\gamma, s) \in \Omega_X^M$ is given by

$$T_\omega(\Omega_X^M) := L^2(X \to T(X) \dotplus \mathfrak{g}; \gamma),$$

where $\dotplus$ means direct sum.

Next, we derive an integration by parts formula on Ω_X^M, that is, we get an explicit formula for the dual operator $\operatorname{div}^\Omega$ of the gradient ∇^Ω on Ω_X^M. We prove that the probability measures on Ω_X^M for which ∇^Ω and $\operatorname{div}^\Omega$ become dual operators (with respect to $\langle \cdot, \cdot \rangle_{T(\Omega_X^M)}$) are exactly the mixed marked Poisson measures

$$\mu_{\varkappa,\widetilde{\sigma}} = \int_{\mathbb{R}_+} \pi_{z\widetilde{\sigma}}\, \varkappa(dz),$$

where $\varkappa$ is a probability measure on $\mathbb{R}_+$ (with finite first moment) and $\pi_{z\widetilde{\sigma}}$ is the marked Poisson measure on Ω_X^M with Lévy measure $z\widetilde{\sigma}$, $z \geq 0$. This means that the mixed marked Poisson measures are exactly the "volume elements" corresponding to our differential geometry on Ω_X^M.

Thus, having identified the right volume elements on Ω_X^M, we introduce for each measure $\mu_{\varkappa,\widetilde{\sigma}}$ the first order Sobolev space $H_0^{1,2}(\Omega_X^M, \mu_{\varkappa,\widetilde{\sigma}})$ by closing the corresponding Dirichlet form

$$\mathcal{E}^\Omega_{\mu_{\varkappa,\widetilde{\sigma}}}(F, G) = \int_{\Omega_X^M} \langle \nabla^\Omega F, \nabla^\Omega G \rangle_{T(\Omega_X^M)}\, d\pi_{\varkappa,\widetilde{\sigma}}, \qquad F, G \in \mathcal{F}C_{\mathrm{b}}^\infty\big(\mathfrak{D}, \Omega_X^M\big),$$

on $L^2(\Omega_X^M, \mu_{\varkappa,\widetilde{\sigma}})$. Just as in the analysis on the usual configuration space, this is the step where we really start doing real infinite dimensional analysis. The corresponding Dirichlet operator is denoted by $H^\Omega_{\mu_{\varkappa,\widetilde{\sigma}}}$; it is a positive definite selfadjoint operator on $L^2(\Omega_X^M, \mu_{\varkappa,\widetilde{\sigma}})$. The heat semigroup $\big(\exp(-tH^\Omega_{\mu_{\varkappa,\widetilde{\sigma}}})\big)_{t\geq 0}$ generated by it is calculated explicitly. The results

on the ergodicity of this semigroup are absolutely analogous to the corresponding results of [7]. Particularly, we have ergodicity if and only if $\mu_{\varkappa,\widetilde{\sigma}} = \pi_{z\widetilde{\sigma}}$ for some $z > 0$, i.e., $\mu_{\varkappa,\widetilde{\sigma}}$ is a (pure) marked Poisson measure.

We also clarify the relation between the intrinsic geometry on Ω_X^M we have constructed with another kind of extrinsic geometry on Ω_X^M which is based on fixing the marked Poisson measure $\pi_{\widetilde{\sigma}}$ and considering the unitary isomorphism between $L^2(\Omega_X^M, \pi_{\widetilde{\sigma}})$ and the corresponding Fock space

$$\mathcal{F}(L^2(X \times M; \widetilde{\sigma})) = \bigoplus_{n=0}^{\infty} \hat{L}^2((X \times M)^n, n!\,\widetilde{\sigma}^{\otimes n}),$$

where $\hat{L}^2((X \times M)^n, n!\,\widetilde{\sigma}^{\otimes n})$ is the subspace of symmetric functions from $L^2((X \times M)^n, n!\,\widetilde{\sigma}^{\otimes n})$. Our main result here is to prove that $H_{\pi_{\widetilde{\sigma}}}^{\Omega}$ is unitarily equivalent (under the above isomorphism) to the second quantization operator of the Dirichlet operator $H_{\widetilde{\sigma}}^{X \times M}$ on the $L^2(X \times M; \widetilde{\sigma})$ space.

As a consequence of the results of this paper, we obtain a representation on the marked Poisson space $L^2(\pi_{\widetilde{\sigma}})$ not only of the group $\mathfrak{A}$, but also of its Lie algebra $\mathfrak{a}$. Let us remark that the groups of smooth (as well as measurable and continuous) currents are classical objects in representation theory, see e.g. [4, 41, 11, 12, 43, 20] and references therein for different representations of these groups. On the other hand, different representations of the group $\mathfrak{A}$ and its Lie algebra $\mathfrak{a}$, in the special case $G = \mathfrak{g} = \mathbb{R}$, were constructed and studied by G. Goldin et al. [17, 19, 16] from the point of view of nonrelativistic quantum mechanics.

Finally, we note that, in a way parallel to the work [8], the results of the present paper can be generalized to the interaction case where, instead of the Poisson measure $\pi_{\widetilde{\sigma}}$, describing a system of free particles, one takes its Gibbsian perturbation—more exactly, a marked Gibbs measure on Ω_X^M of Ruelle type (see [28, 29]).

1 Marked Poisson measures

1.1 Marked configuration space

Let X be a connected, oriented C^∞ non-compact Riemannian manifold. The configuration space Γ_X over X is defined as the set of all locally finite subsets in X:

$$\Gamma_X := \big\{\, \gamma \subset X \mid \#(\gamma \cap K) < \infty \text{ for each compact } K \subset X \,\big\},$$

where $\#(\cdot)$ denotes the cardinality of a set. One can identify any $\gamma \in \Gamma_X$ with the positive integer-valued Radon measure

$$\sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}(X),$$

where $\sum_{x \in \varnothing} \varepsilon_x :=$ zero measure and $\mathcal{M}(X)$ denotes the set of all positive Radon measures on $\mathcal{B}(X)$.

Let also M be a connected oriented C^∞ (compact or non-compact) Riemannian manifold. The marked configuration space Ω_X^M over X with marks from M is defined as

$$\Omega_X^M := \big\{ \omega = (\gamma, s) \mid \gamma \in \Gamma_X,\ s \in M^\gamma \big\},$$

where M^γ stands for the set of all maps $\gamma \ni x \mapsto m \in M$. Equivalently, we can define Ω_X^M as the collection of subsets in $X \times M$ having the following properties:

$$\Omega_X^M = \left\{ \omega \subset X \times M \;\middle|\; \begin{array}{c} \text{a) } \forall (x,m),(x',m') \in \omega : (x,m) \neq (x',m') \Rightarrow x \neq x' \\ \text{b) } \mathrm{Pr}_X\, \omega \in \Gamma_X \end{array} \right\},$$

where Pr_X denotes the projection of the Cartesian product of X and M onto X. Again, each $\omega \in \Omega_X^M$ can be identified with the measure

$$\sum_{(x,m)\in\omega} \varepsilon_{(x,m)} \in \mathcal{M}(X \times M).$$

It is worth noting that, for any bijection $\phi\colon X \times M \to X \times M$, the image of the measure $\omega(\cdot)$ under the mapping ϕ, $(\phi^*\omega)(\cdot)$, coincides with $(\phi(\omega))(\cdot)$, i.e.,

$$(\phi^*\omega)(\cdot) = (\phi(\omega))(\cdot), \qquad \omega \in \Omega_X^M,$$

where $\phi(\omega) = \{\phi(x,m) \mid (x,m) \in \omega\}$ is the image of ω as a subset of $X \times M$.

Let $\mathcal{B}_\mathrm{c}(X)$ and $\mathcal{O}_\mathrm{c}(X)$ denote the families of all Borel, resp. open subsets of X that have compact closure. Let also $\mathcal{B}_\mathrm{c}(X \times M)$ denote the family of all Borel subsets of $X \times M$ whose projection on X belongs to $\mathcal{B}_\mathrm{c}(X)$.

Denote by $C_{0,\mathrm{b}}(X \times M)$ the set of real-valued bounded continuous functions f on $X \times M$ such that $\operatorname{supp} f \in \mathcal{B}_\mathrm{c}(X \times M)$. As usually, we set for any $f \in C_{0,\mathrm{b}}(X \times M)$ and $\omega \in \Omega_X^M$

$$\langle f, \omega \rangle = \int_{X\times M} f(x,m)\, \omega(dx,dm) = \sum_{(x,m)\in\omega} f(x,m).$$

We note that, because of the definition of Ω_X^M, there are only a finite number of addends in the latter series.

Now, we are going to discuss the measurable structure of the space Ω_X^M. We will use a "localized" description of the Borel σ-algebra $\mathcal{B}(\Omega_X^M)$ over Ω_X^M.

For $\Lambda \in \mathcal{O}_\mathrm{c}(X)$, define

$$\Omega_\Lambda^M := \big\{ \omega \in \Omega_X^M \mid \mathrm{Pr}_X\, \omega \subset \Lambda \big\}$$

and for $n \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$

$$\Omega_\Lambda^M(n) := \big\{ \omega \in \Omega_\Lambda^M \mid \#(\omega) = n \big\}.$$

It is obvious that

$$\Omega_\Lambda^M = \bigsqcup_{n=0}^\infty \Omega_\Lambda^M(n).$$

Let $\Lambda_{\mathrm{mk}} := \Lambda \times M$ (i.e., Λ_{mk} is the set of all "marked" elements of Λ) and let

$$\widetilde{\Lambda}^n_{\mathrm{mk}} := \big\{((x_1,m_1),\dots,(x_n,m_n)) \in \Lambda^n_{\mathrm{mk}} \mid x_j \neq x_k \text{ if } j \neq k\,\big\}.$$

There is a bijection

$$\mathcal{L}^{(n)}_\Lambda \colon \widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n \mapsto \Omega^M_\Lambda(n) \tag{1.1}$$

given by

$$\mathcal{L}^{(n)}_\Lambda \colon ((x_1,m_1),\dots,(x_n,m_n)) \mapsto \{(x_1,m_1),\dots,(x_n,m_n)\} \in \Omega^M_\Lambda(n),$$

where $\mathfrak{S}_n$ is the permutation group over $\{1,\dots,n\}$. On $\Lambda^n_{\mathrm{mk}}/\mathfrak{S}_n$ one introduces the related metric

$$\begin{aligned}\delta&\big[((x_1,m_1),\dots,(x_n,m_n)),((x_1',m_1'),\dots,(x_n',m_n'))\big]\\ &= \inf_{\sigma\in\mathfrak{S}_n} d^n\big[((x_1,m_1),\dots,(x_n,m_n)),((x'_{\sigma(1)},m'_{\sigma(1)}),\dots,(x'_{\sigma(n)},m'_{\sigma(n)}))\big],\end{aligned}$$

where d^n is the metric on Λ^n_{mk} driven from the original metrics on X and M. Then, $\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n$ becomes an open set in $\Lambda^n_{\mathrm{mk}}/\mathfrak{S}_n$ and let $\mathcal{B}(\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n)$ be the trace σ-algebra on $\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n$ generated by $\mathcal{B}(\Lambda^n_{\mathrm{mk}}/\mathfrak{S}_n)$. Let then $\mathcal{B}(\Omega^M_\Lambda(n))$ be the image σ-algebra of $\mathcal{B}(\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n)$ under the bijection $\mathcal{L}^{(n)}_\Lambda$ and let $\mathcal{B}(\Omega^M_\Lambda)$ be the σ-algebra on Ω^M_Λ generated by the usual topology of (disjoint) union of topological spaces.

For any $\Lambda \in \mathcal{O}_{\mathrm{c}}(X)$, there is a natural restriction map $p_\Lambda \colon \Omega^M_X \mapsto \Omega^M_\Lambda$ defined by

$$\Omega^M_X \ni \omega \mapsto p_\Lambda(\omega) := \omega \cap \Lambda_{\mathrm{mk}} \in \Omega^M_\Lambda.$$

The topology on Ω^M_X is defined as the weakest topology making all the mappings p_Λ continuous. The associated σ-algebra is denoted by $\mathcal{B}(\Omega^M_X)$.

For each $B \in \mathcal{B}_{\mathrm{c}}(X\times M)$, we introduce a function $N_B \colon \Omega^M_X \to \mathbb{Z}_+ = \{0,1,2,\dots\}$ such that

$$N_B(\omega) := \#(\omega \cap B), \qquad \omega \in \Omega^M_X. \tag{1.2}$$

Then, it is not hard to see that $\mathcal{B}(\Omega^M_X)$ is the smallest σ-algebra on Ω^M_X such that all the functions N_B are measurable.

1.2 Marked Poisson measure

In order to construct a marked Poisson measure, we fix:

(i) an intensity measure σ on the underlying manifold X, which is supposed to be a nonatomic Radon one,

(ii) a non-negative function

$$X \times \mathcal{B}(M) \ni (x,\Delta) \mapsto p(x,\Delta) \in \mathbb{R}_+$$

such that, for σ-a.a. $x \in X$, $p(x,\cdot)$ is a finite measure on M.

Now, we define a measure $\widetilde{\sigma}$ on $(X \times M, \mathcal{B}(X \times M))$ as follows:

$$\widetilde{\sigma}(A) = \int_A p(x, dm)\, \sigma(dx), \qquad A \in \mathcal{B}(X \times M). \tag{1.3}$$

We will suppose that the measure $\widetilde{\sigma}$ is infinite and for any $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$

$$\widetilde{\sigma}(\Lambda_{\mathrm{mk}}) = \int_X \mathbf{1}_\Lambda(x) p(x, M)\, \sigma(dx) < \infty, \tag{1.4}$$

i.e., $p(x, M) \in L^1_{\mathrm{loc}}(\sigma)$.

Now, we wish to introduce a marked Poisson measure on Ω_X^M (cf. e.g. [23, 22]). To this end, we take first the measure $\widetilde{\sigma}^{\otimes n}$ on $(X \times M)^n$, and for any $\Lambda \in \mathcal{O}_{\mathrm{c}}(X)$, $\widetilde{\sigma}^{\otimes n}$ can be considered as a finite measure on Λ^n_{mk}. Since σ is nonatomic, we get

$$\widetilde{\sigma}^{\otimes n}(\Lambda^n_{\mathrm{mk}} \setminus \widetilde{\Lambda}^n_{\mathrm{mk}}) = 0$$

and we can consider $\widetilde{\sigma}^{\otimes n}$ as a measure on $(\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n, \mathcal{B}(\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n))$ such that

$$\widetilde{\sigma}^{\otimes n}(\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n) = \widetilde{\sigma}(\Lambda_{\mathrm{mk}})^n.$$

Denote by $\widetilde{\sigma}_{\Lambda,n} := \widetilde{\sigma}^{\otimes n} \circ (\mathcal{L}^{(n)}_\Lambda)^{-1}$ the image measure on $\Omega^M_\Lambda(n)$ under the bijection (1.1). Then, we can define a measure $\lambda^\Lambda_{\widetilde{\sigma}}$ on Ω^M_Λ by

$$\lambda^\Lambda_{\widetilde{\sigma}} := \sum_{n=0}^\infty \frac{1}{n!}\, \widetilde{\sigma}_{\Lambda,n},$$

where $\widetilde{\sigma}_{\Lambda,0} := \varepsilon_\varnothing$ on $\Omega^M_\Lambda(0) = \{\varnothing\}$. The measure $\lambda^\Lambda_{\widetilde{\sigma}}$ is finite and $\lambda^\Lambda_{\widetilde{\sigma}}(\Omega^M_\Lambda) = e^{\widetilde{\sigma}(\Lambda_{\mathrm{mk}})}$. Hence, the measure

$$\pi^\Lambda_{\widetilde{\sigma}} := e^{-\widetilde{\sigma}(\Lambda_{\mathrm{mk}})} \lambda^\Lambda_{\widetilde{\sigma}}$$

is a probability measure on $\mathcal{B}(\Omega^M_\Lambda)$. It is not hard to check the consistency property of the family $\{\pi^\Lambda_{\widetilde{\sigma}} \mid \Lambda \in \mathcal{O}_{\mathrm{c}}(X)\}$ and thus to obtain a unique probability measure $\pi_{\widetilde{\sigma}}$ on $\mathcal{B}(\Omega^M_X)$ such that

$$\pi^\Lambda_{\widetilde{\sigma}} = p^*_\Lambda \pi_{\widetilde{\sigma}}, \qquad \Lambda \in \mathcal{O}_{\mathrm{c}}(X).$$

This measure $\pi_{\widetilde{\sigma}}$ will be called a marked Poisson measure with Lévy measure $\widetilde{\sigma}$.

For any function $\varphi \in C_{0,\mathrm{b}}(X \times M)$, it is easy to calculate the Laplace transform of the measure $\pi_{\widetilde{\sigma}}$

$$\ell_{\pi_{\widetilde{\sigma}}}(\varphi) := \int_{\Omega^M_X} e^{\langle \varphi, \omega\rangle}\, \pi_{\widetilde{\sigma}}(d\omega) = \exp\bigg(\int_{X \times M} (e^{\varphi(x,m)} - 1)\, \widetilde{\sigma}(dx, dm)\bigg). \tag{1.5}$$

Example 1.1 Let $p(x, \cdot) \equiv \varepsilon_m(\cdot)$, where m is some fixed point of M and $x \in X$. Then, $\widetilde{\sigma} = \sigma \otimes \varepsilon_m$ and $\pi_{\widetilde{\sigma}} = \pi_\sigma$ is just the Poisson measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ with intensity σ.

Example 1.2 Let $p(x, \cdot) \equiv \tau(\cdot)$, $x \in X$, where τ is a finite measure on $(M, \mathcal{B}(M))$. Now, $\widetilde{\sigma} = \hat{\sigma} = \sigma \otimes \tau$ and $\pi_{\widetilde{\sigma}}$ coincides with the marked Poisson measure under consideration in [26] (in the case where M is a manifold). Notice that the choice of $\widetilde{\sigma} = \hat{\sigma}$ as a product measure means a position-independent marking, while the choice of a general $\widetilde{\sigma}$ of the form (1.3) leads to a position-depending marking.

2 Transformations of the marked Poisson measure

2.1 Group of transformations of the marked configuration space

We are looking for a natural group $\mathfrak{A}$ of transformations of Ω_X^M such that

(i) $\pi_{\widetilde{\sigma}}$ is $\mathfrak{A}$-quasiinvariant;

(ii) $\mathfrak{A}$ is big enough to reconstruct $\pi_{\widetilde{\sigma}}$ by the Radon–Nikodym density $\dfrac{da^*\pi_{\widetilde{\sigma}}}{d\pi_{\widetilde{\sigma}}}$, where a runs through $\mathfrak{A}$.

Let us recall that in the work [26] the group $\mathrm{Diff}_0(X)$ was taken as $\mathfrak{A}$, just in the same way as in the case of the usual Poisson measure [7]. Here, $\mathrm{Diff}_0(X)$ stands for the group of diffeomorphisms of X with compact support, i.e., each $\psi \in \mathrm{Diff}_0(X)$ is a diffeomorphism of X that is equal to the identity outside a compact set (depending on ψ). The group $\mathrm{Diff}_0(X)$ satisfies (i). However, unlike the case of the Poisson measure, the condition (ii) is not satisfied, because, for example, in the case where $\widetilde{\sigma} = \sigma \otimes \tau$, there is no information about the measure τ that is contained in $\dfrac{d\psi^*\pi_{\widetilde{\sigma}}}{d\pi_{\widetilde{\sigma}}}$, see [26]. Therefore, just as in the case of [24], we need a proper extension of the group $\mathrm{Diff}_0(X)$.

In what follows, we will suppose that M is a homogeneous space of a Lie group G (see e.g. [10]). Let us recall that this means the existence of a C^∞ mapping $\theta\colon G \times M \to M$ satisfying the following conditions:

(i) If e is the unity element of the group G, then

$$\theta(e, m) = m \qquad \text{for all } m \in M;$$

(ii) If $g_1, g_2 \in G$, then

$$\theta(g_1, \theta(g_2, m)) = \theta(g_1 g_2, m) \qquad \text{for all } m \in M;$$

(iii) For arbitrary $m_1, m_2 \in M$, there exists $g \in G$ such that $\theta(g, m_1) = m_2$.

For any $g \in G$, we will denote by $\theta_g\colon M \to M$ the mapping given by $\theta_g(m)\colon = \theta(g, m)$; then θ_g defines a diffeomorphism of M.

Let us fix an arbitrary point $m_0 \in M$ and let H be the isotropy group of M:

$$H := \big\{\, g \in G \mid \theta_g(m_0) = m_0 \,\big\}.$$

Then, the homogeneous space M can always be identified with the factor space G/H (endowed with the unique corresponding C^∞ manifold structure), i.e., $M = G/H$.

Let us consider the group of *smooth currents*, i.e., all C^∞ mappings $X \ni x \mapsto \eta(x) \in G$, which are equal to e outside a compact set (depending on η). A multiplication $\eta_1\eta_2$ in this group is defined as the pointwise multiplication of the mappings η_1 and η_2. In the representation theory this group is denoted by G^X, or $C_0^\infty(X; G)$.

The group $\mathrm{Diff}_0(X)$ acts in G^X by automorphisms: for each $\psi \in \mathrm{Diff}_0(X)$,

$$G^X \ni \eta \overset{\alpha}{\mapsto} \alpha(\psi)\eta := \eta \circ \psi^{-1} \in G^X.$$

Thus, we can endow the Cartesian product of $\mathrm{Diff}_0(X)$ and G^X with the following multiplication: for $a_1 = (\psi_1, \eta_1)$, $a_2 = (\psi_2, \eta_2)$ from $\mathrm{Diff}_0(X) \times G^X$

$$a_1 a_2 = (\psi_1 \circ \psi_2, \eta_1(\eta_2 \circ \psi_1^{-1}))$$

and obtain a semidirect product

$$\mathrm{Diff}_0(X) \underset{\alpha}{\times} G^X =: \mathfrak{A}$$

of the groups $\mathrm{Diff}_0(X)$ and G^X.

The group $\mathfrak{A}$ acts in $X \times M$ in the following way: for any $a = (\psi, \eta) \in \mathfrak{A}$

$$X \times M \ni (x, m) \mapsto a(x, m) = (\psi(x), \theta(\eta(\psi(x)), m)) \in X \times M. \tag{2.1}$$

If id denotes the identity diffeomorphism of X and $\boldsymbol{e}$ is the function identically equal to e on X, then we will just identify ψ with $(\psi, \boldsymbol{e})$ and η with (id, η). The action (2.1) of an arbitrary $a = (\psi, \eta)$ can be represented as

$$(x, m) \mapsto a(x, m) = \eta\psi(x, m),$$

where

$$\begin{aligned} \psi(x, m) &= (\psi(x), m), \\ \eta(x, m) &= (x, \theta(\eta(x), m)). \end{aligned}$$

For any $a = (\psi, \eta) \in \mathfrak{A}$, denote $K_a := K_\psi \cup K_\eta$, where K_ψ and K_η are the minimal closed sets in X outside of which $\psi = \mathrm{id}$ and $\eta = \boldsymbol{e}$, respectively. Evidently, $K_a \in \mathcal{B}_{\mathrm{c}}(X)$,

$$a(K_a)_{\mathrm{mk}} = (K_a)_{\mathrm{mk}},$$

and a is the identity transformation outside $(K_a)_{\mathrm{mk}}$.

Now, let us recall some known facts concerning quasiinvariant measures on homogeneous spaces (see e.g. [45, 44]).

Theorem 2.1 *Suppose G is a Lie group and H its subgroup, and let dg, δ_G and dh, δ_H be fixed Haar measures and modular functions on G and H, respectively. Then:*

(i) *for every measure μ on G/H that is quasiinvariant with respect to the action of G on G/H, there exists a measurable positive function ξ on G verifying*

$$\xi(gh) = \frac{\delta_H(h)}{\delta_G(h)}\, \xi(g), \qquad g \in G,\ h \in H, \tag{2.2}$$

and
$$\int_G f(g)\xi(g)\,dg = \int_{G/H} \mu(d\,gH) \int_H f(gh)\,dh, \qquad f \in C_0(G), \tag{2.3}$$
where $C_0(G)$ denotes the set of continuous functions on G with compact support; for each $g \in G$ the Radon–Nikodym density is given by
$$p_g^{\mu}(\widetilde{g}H) := \frac{dg^*\mu}{d\mu}(\widetilde{g}H) = \frac{\xi(g^{-1}\widetilde{g})}{\xi(\widetilde{g})}, \qquad \widetilde{g}H \in G/H;$$

(ii) *there exists a quasiinvariant measure λ on G/H such that the function*
$$p^{\lambda}(g, \widetilde{g}H) := p_g^{\lambda}(\widetilde{g}H)$$
is differentiable on $G \times G/H$.

Remark 2.1 We recall that the modular function $\delta_G(\cdot)$ of a Lie group G is defined from the equality $r_{\widetilde{g}}^*\,dg = \delta_G(\widetilde{g})\,dg$, where dg is the Haar measure on G (i.e., a fixed left-invariant measure on G) and r_g denotes the right translation on G, i.e., $\widetilde{g} \mapsto r_g\widetilde{g} = g\widetilde{g}$.

We fix the measure λ on $M = G/H$ from Theorem 2.1, (ii). As easily seen from Theorem 2.1 (i), any quasiinvariant measure on M in equivalent to λ.

Remark 2.2 If $H = \{e\}$, i.e., $M = G$, then we can choose λ to be the Haar measure dg on G. Moreover, if $\delta_G(h) = \delta_H(h)$ for all $h \in H$ (and only in this case) there exists a λ being invariant with respect to the action of G on M. The latter condition holds automatically if G is unimodular, that is, $\delta_G(g) \equiv 1$ for all $g \in G$. This, in turn, holds for all compact and simple Lie groups.

In what follows, we will suppose that the measure σ is equivalent to the Riemannian volume ν on X: $\sigma(dx) = \rho(x)\,\nu(dx)$ with $\rho > 0$ ν-a.s., and that for ν-a.a. $x \in X$ $p(x, \cdot)$ is equivalent to the measure λ:
$$p(x, dm) = p(x, m)\,\lambda(dm) \quad \text{with } p(x, m) > 0 \ \lambda\text{-a.a. } m \in M.$$
Thus, the measure $\widetilde{\sigma}$ can be written in the form
$$\widetilde{\sigma}(dx, dm) = \rho(x)p(x, m)\,\nu(dx)\,\lambda(dm).$$
The condition $\widetilde{\sigma}(\Lambda_{\mathrm{mk}}) < \infty$, $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$, implies that the function
$$q(x, m) := \rho(x)p(x, m)$$
satisfies
$$q^{1/2} \in L^2_{\mathrm{loc}}(X; \nu) \otimes L^2(M; \lambda). \tag{2.4}$$

Noting that
$$a^{-1}(x, m) = (\psi, \eta)^{-1}(x, m) = (\psi^{-1}(x), \theta(\eta^{-1}(x), m)),$$
we easily deduce the following

Proposition 2.1 *The measure $\widetilde{\sigma}$ is $\mathfrak{A}$-quasiinvariant and for any $a = (\psi, \eta) \in \mathfrak{A}$ the Radon–Nikodym density is given by*

$$\begin{cases} p_a^{\widetilde{\sigma}}(x,m) := \dfrac{d(a^*\widetilde{\sigma})}{d\widetilde{\sigma}}(x,m) = \dfrac{q(\psi^{-1}(x), \theta(\eta^{-1}(x), m))}{q(x,m)} p^\lambda(\eta(x), m)\, J_\nu^\psi(x), \\ \text{if } (x,m) \in \{0 < q(x,m) < \infty\} \cap \{0 < q(\psi^{-1}(x), \theta(\eta^{-1}(x), m)) < \infty\}, \\ p_a^{\widetilde{\sigma}}(x,m) = 1, \qquad \text{otherwise,} \end{cases}$$

where J_ν^ψ is the Jacobian determinant of ψ (w.r.t. the Riemannian volume ν).

We give two examples of the above construction, which are important from the point of view of the marked configuration space analysis. We refer the reader to e.g. [44, 45] for further examples.

Example 2.1 Let $G = \mathbb{R}_+$ be the dilation group (e.g. [15]), i.e., the multiplication in this group is given by the usual multiplication of numbers. As a homogeneous space M we take G itself, by identifying the action of the group with the multiplication in it. As a quasiinvariant measure λ on M we can take the restriction to $\mathbb{R}_+$ of the Lebesgue measure on $\mathbb{R}$.

The analysis and geometry on the marked configuration space $\Omega_X^{\mathbb{R}_+}$ were studied in our previous work [24]. Here we only mention that the choice $M = \mathbb{R}_+$ leads (via a natural isomorphism) to the class of compound Poisson measures. In other words, each mark $s_x \in \mathbb{R}_+$ corresponding to $x \in X$ describes the charge of the measure

$$\omega = (\gamma, s) = \sum_{x \in X} s_x \varepsilon_x \in \mathcal{M}(X)$$

at the point x (or, in the case where $X = \mathbb{R}$, the value of the jump of the process at x).

Example 2.2 Let $G = O(d+1)$ be the $(d+1)$-dimensional orthogonal group and let $M = S^d$ be the d-dimensional unit sphere in $\mathbb{R}^{d+1}$ with the natural action of the group $O(d+1)$ on S^d, see e.g. [13, 44, 45]. As λ we take the surface measure on S^d, which is invariant w.r.t. the action of $O(d+1)$. From the point of view of statistical mechanics, a mark $s_x \in S^d$ describes in this example the spin of the particle at the point x.

2.2 $\mathfrak{A}$-quasiinvariance of the marked Poisson measure

Any $a \in \mathfrak{A}$ defines by (2.1) a transformation of $X \times M$, and, consequently, a has the following "lifting" from $X \times M$ to Ω_X^M:

$$\Omega_X^M \ni \omega \mapsto a(\omega) = \left\{\, a(x,m) \mid (x,m) \in \omega \,\right\} \in \Omega_X^M. \tag{2.5}$$

(Note that, for a given $\omega \in \Omega_X^M$, $a(\omega)$ indeed belongs to Ω_X^M and coincides with ω for all but a finite number of points.) The mapping (2.5) is obviously measurable and we can define the image $a^*\pi_{\widetilde{\sigma}}$ as usually. The following proposition is an analog of a corresponding fact about Poisson measures.

Proposition 2.2 *For any $a \in \mathfrak{A}$, we have*

$$a^*\pi_{\tilde{\sigma}} = \pi_{a^*\tilde{\sigma}}.$$

Proof. The proof is the same as for the usual Poisson measure π_σ with intensity σ and $\psi \in \mathrm{Diff}_0(X)$ (e.g., [7]), one has just to calculate the Laplace transform of the measure $a^*\pi_{\tilde{\sigma}}$ for any $f \in C_{0,\mathrm{b}}(X \times M)$ and to use the formula (1.5). ■

Proposition 2.3 *The marked Poisson measure $\pi_{\tilde{\sigma}}$ is quasiinvariant w.r.t. the group $\mathfrak{A}$, and for any $a \in \mathfrak{A}$ we have*

$$\frac{d(a^*\pi_{\tilde{\sigma}})}{d\pi_{\tilde{\sigma}}}(\omega) = \prod_{(x,m)\in\omega} p_a^{\tilde{\sigma}}(x,m). \tag{2.6}$$

Proof. The result follows from Skorokhod theorem on absolute continuity of Poisson measures (see, e.g., [39, 40]). ■

Remark 2.3 Notice that only a finite (depending on ω) number of factors in the product on the right hand side of (2.6) are not equal to one.

3 The differential geometry of marked configuration spaces

3.1 The tangent bundle of Ω_X^M

Let us denote by $V_0(X)$ the set of C^∞ vector fields on X (i.e., smooth sections of $T(X)$) that have compact support. Let $\mathfrak{g}$ denote the Lie algebra of G and let $C_0^\infty(X;\mathfrak{g})$ stand for the set of all C^∞ mappings of X into $\mathfrak{g}$ that have compact support. Then

$$\mathfrak{a} := V_0(X) \times C_0^\infty(X;\mathfrak{g})$$

can be thought of as a Lie algebra corresponding to the Lie group $\mathfrak{A}$. More precisely, for any fixed $v \in V_0(X)$ and for any $x \in X$, the curve

$$\mathbb{R} \ni t \mapsto \psi_t^v(x) \in X$$

is defined as the solution of the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\psi_t^v(x) = v(\psi_t^v(x)), \\ \quad \psi_0^v(x) = x. \end{cases} \tag{3.1}$$

Then, the mappings $\{\psi_t^v,\, t \in \mathbb{R}\}$ form a one-parameter subgroup of diffeomorphisms in $\mathrm{Diff}_0(X)$ (see, e.g., [10]):

$$\begin{aligned} &1)\forall t \in \mathbb{R} \quad \psi_t^v \in \mathrm{Diff}_0(X), \\ &2)\forall t_1, t_2 \in \mathbb{R} \quad \psi_{t_1}^v \circ \psi_{t_2}^v = \psi_{t_1+t_2}^v. \end{aligned}$$

Next, for each function $u \in C_0^\infty(X;\mathfrak{g})$, $x \in X$, and $t \in \mathbb{R}$, we set $\eta_t^u(x) := \exp(tu(x))$, where $\mathfrak{g} \ni Y \mapsto \exp Y \in G$ is the exponential mapping (see, e.g., [45]). Hence, for a fixed $x \in X$, $\{\eta_t^u(x),\ t \in \mathbb{R}\}$ is a one-parameter subgroup of G and

$$\begin{aligned} \eta_0^u(x) &= e, \\ \frac{d}{dt}\,\eta_t^u(x)\big|_{t=0} &= u(x). \end{aligned} \tag{3.2}$$

Let us recall a fundamental theorem in the theory of Lie groups.

Theorem 3.1 *There exists a neighborhood U of the zero in $\mathfrak{g}$ and a neighborhood O of the unit element e in G such that* $\exp\colon U \to O$ *is an analytic diffeomorphism.*

From this theorem, we conclude that, for each fixed $u \in C_0^\infty(X;\mathfrak{g})$, there exists $\varepsilon > 0$ such that for any $t \in (-\varepsilon,\varepsilon)$ the mapping $X \ni x \mapsto \eta_t^u(x) \in G$ belongs to G^X, which yields, in turn, that $\eta_t^u \in G^X$ for all $t \in \mathbb{R}$, and moreover η_t^u is a one-parameter subgroup of G^X.

Thus, for an arbitrary $(v,u) \in \mathfrak{a}$, we can consider the curve $\{(\psi_t^v,\eta_t^u),\ t \in \mathbb{R}\}$ in $\mathfrak{A}$. Hence, to any $\omega \in \Omega_X^M$ there corresponds the following curve in Ω_X^M:

$$\mathbb{R} \ni t \mapsto (\psi_t^v,\eta_t^u)\omega \in \Omega_X^M.$$

Define now for a function $F\colon \Omega_X^M \to \mathbb{R}$ the directional derivative of F along (v,u) as

$$(\nabla_{(v,u)}^\Omega F)(\omega) := \frac{d}{dt}F((\psi_t^v,\eta_t^u)\omega)\big|_{t=0},$$

provided the right hand side exists. We will also denote by ∇_v^Ω and ∇_u^Ω the directional derivatives along $(v,0)$ and $(0,u)$, respectively.

Absolutely analogously, one defines for a function $\varphi\colon X \times M \to \mathbb{R}$ the directional derivative of φ along (v,u):

$$(\nabla_{(v,u)}^{X\times M}\varphi)(x,m) = \frac{d}{dt}\varphi((\psi_t^v,\eta_t^u)(x,m))\big|_{t=0}. \tag{3.3}$$

Then, for a continuously differentiable function φ, we have from (2.1), (3.1), (3.2), and (3.3)

$$\begin{aligned} (\nabla_{(v,u)}^{X\times M}\varphi)(x,m) &= \frac{d}{dt}\,\varphi((\psi_t^v(x),\theta(\eta_t^u(\psi_t^v(x)),m)\big|_{t=0} \\ &= \frac{d}{dt}\varphi(\psi_t^v(x),m)\big|_{t=0} + \frac{d}{dt}\varphi(x,\theta(\eta_t^u(x),m))\big|_{t=0} \\ &\quad + \frac{d}{dt}\varphi(x,\theta(\eta_0^u(\psi_t^v(x)),m))\big|_{t=0} \\ &= \langle \nabla^X\varphi(x,m),v(x)\rangle_{T_x(X)} + \langle \nabla^G\varphi(x,\theta(e,m)),u(x)\rangle_{\mathfrak{g}} \\ &= \langle \nabla^{X\times M}\varphi(x,m),(v(x),u(x))\rangle_{T_{(x,m)}(X\times M)}. \end{aligned} \tag{3.4}$$

Here, $T_{(x,m)}(X \times M) := T_x(X) \dotplus \mathfrak{g}$ and $\nabla^{X\times M} := (\nabla^X, \widetilde{\nabla}^M)$, where ∇^X denotes the gradient on X and

$$\begin{aligned} \widetilde{\nabla}^M f(m) &= \nabla^G \hat{f}(e,m), \\ \hat{f}(g,m) &:= f(\theta(g,m)), \qquad g \in G,\ m \in M, \end{aligned} \tag{3.5}$$

∇^G being the gradient on G.

Remark 3.1 Notice that upon (3.5) we have, for a fixed $u \in \mathfrak{g}$,

$$\begin{aligned} \langle \widetilde{\nabla}^M f(m), u\rangle_{\mathfrak{g}} &= \langle \nabla^G f(\theta(e,m)), u\rangle_{\mathfrak{g}} \\ &= \frac{d}{dt} f(\theta(e^{tu}, m))\big|_{t=0} \\ &= \langle \nabla^M f(m), (Ru)(m)\rangle_{T_m(M)}, \end{aligned} \tag{3.6}$$

where ∇^M denotes the usual gradient on M, and the vector field Ru on M is given by

$$M \ni m \mapsto (Ru)(m) := \frac{d}{dt}\theta(e^{tu}, m)\big|_{t=0}. \tag{3.7}$$

Let us introduce a special class of "nice functions" on Ω_X^M. Denote by $\mathfrak{D}$ the set of all C^∞-functions φ on $X \times M$ such that the support of φ is in $\mathcal{B}_{\mathrm{c}}(X \times M)$, and φ and all its $\nabla^{X\times M}$ derivatives are bounded. Next, let $C_{\mathrm{b}}^\infty(\mathbb{R}^N)$ stand for the space of all C^∞-functions on $\mathbb{R}^N$ which together with all their derivatives are bounded. Then, we can introduce $\mathcal{F}C_{\mathrm{b}}^\infty\big(\mathfrak{D}, \Omega_X^M\big)$ as the set of all functions $F\colon \Omega_X^M \mapsto \mathbb{R}$ of the form

$$F(\omega) = g_F(\langle \varphi_1, \omega\rangle, \dots, \langle \varphi_N, \omega\rangle), \qquad \omega \in \Omega_X^M, \tag{3.8}$$

where $\varphi_1, \dots, \varphi_N \in \mathfrak{D}$ and $g_F \in C_{\mathrm{b}}^\infty(\mathbb{R}^N)$ (compare with [7]). $\mathcal{F}C_{\mathrm{b}}^\infty\big(\mathfrak{D}, \Omega_X^M\big)$ will be called the set of smooth cylinder functions on Ω_X^M.

For any $F \in \mathcal{F}C_{\mathrm{b}}^\infty\big(\mathfrak{D}, \Omega_X^M\big)$ of the form (3.8) and a given $(v,u) \in \mathfrak{a}$, we have, just as in [7],

$$\begin{aligned} F((\psi_t^v, \eta_t^u)\omega) &= g_F(\langle \varphi_1, (\psi_t^v, \eta_t^u)\omega\rangle, \dots, \langle \varphi_N, (\psi_t^v, \eta_t^u)\omega\rangle) \\ &= g_F(\langle \varphi_1 \circ (\psi_t^v, \eta_t^u), \omega\rangle, \dots, \langle \varphi_N \circ (\psi_t^v, \eta_t^u), \omega\rangle), \end{aligned}$$

and therefore

$$(\nabla_{(v,u)}^\Omega F)(\omega) = \sum_{j=1}^N \frac{\partial g_F}{\partial r_j}(\langle \varphi_1, \omega\rangle, \dots, \langle \varphi_N, \omega\rangle)\langle \nabla_{(v,u)}^{X\times M} \varphi_j, \omega\rangle. \tag{3.9}$$

In particular, we conclude from (3.9) that

$$\nabla_{(v,u)}^\Omega = \nabla_v^\Omega + \nabla_u^\Omega. \tag{3.10}$$

The expression of $\nabla_{(v,a)}^\Omega$ on smooth cylinder functions motivates the following definition.

Definition 3.1 The tangent space $T_\omega(\Omega_X^M)$ to the marked configuration space Ω_X^M at a point $\omega = (\gamma, s) \in \Omega_X^M$ is defined as the Hilbert space

$$\begin{aligned} T_\omega(\Omega_X^M) &:= L^2(X \to T(X) \dotplus \mathfrak{g}; \gamma) \\ &= L^2(X \to T(X); \gamma) \oplus L^2(X \to \mathfrak{g}; \gamma) \\ &= \bigoplus_{x \in \gamma} \big[T_x(X) \oplus \mathfrak{g}\big] \end{aligned}$$

with scalar product

$$\begin{aligned} \langle V_\omega^1, V_\omega^2 \rangle_{T_\omega(\Omega_X^M)} &= \int_X \big(\langle V_\omega^1(x)_{T_x(X)}, V_\omega^2(x)_{T_x(X)} \rangle_{T_x(X)} + \langle V_\omega^1(x)_{\mathfrak{g}}, V_\omega^2(x)_{\mathfrak{g}} \rangle_{\mathfrak{g}}\big)\, \gamma(dx) \\ &= \sum_{x \in \gamma} \big(\langle V_\omega^1(x)_{T_x(X)}, V_\omega^2(x)_{T_x(X)} \rangle_{T_x(X)} + \langle V_\omega^1(x)_{\mathfrak{g}}, V_\omega^2(x)_{\mathfrak{g}} \rangle_{\mathfrak{g}}\big), \qquad (3.11) \end{aligned}$$

where $V_\omega^1, V_\omega^2 \in T_\omega(\Omega_X^M)$ and $V_\omega(x)_{T_x(X)}$ and $V_\omega(x)_{\mathfrak{g}}$ denote the projection of $V_\omega(x) \in T_x(X) \dotplus \mathfrak{g}$ onto $T_x(X)$ and $\mathfrak{g}$, respectively. (Notice that the tangent space $T_\omega(\Omega_X^M)$ depends only on the γ coordinate of ω.) The corresponding tangent bundle is

$$T(\Omega_X^M) = \bigcup_{\omega \in \Omega_X^M} T_\omega(\Omega_X^M).$$

As usually in Riemannian geometry, having directional derivatives and a Hilbert space as a tangent space, we can introduce a gradient.

Definition 3.2 We define the intrinsic gradient ∇^Ω of a function $F \colon \Omega_X^M \to \mathbb{R}$ as the mapping

$$\Omega_X^M \ni \omega \mapsto (\nabla^\Omega F)(\omega) \in T_\omega(\Omega_X^M)$$

such that, for any $(v, u) \in \mathfrak{a}$,

$$(\nabla^\Omega_{(v,u)} F)(\omega) = \langle (\nabla^\Omega F)(\omega), (v, u) \rangle_{T_\omega(\Omega_X^M)}.$$

By (3.9) and (3.4) we have, for an arbitrary $F \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D}, \Omega_X^M)$ of the form (3.8) and each $\omega = (\gamma, s) \in \Omega_X^M$,

$$(\nabla^\Omega F)(\omega; x) = \sum_{j=1}^N \frac{\partial g_F}{\partial r_j}(\langle \varphi_1, \omega \rangle, \dots, \langle \varphi_N, \omega \rangle) \nabla^{X \times M} \varphi_j(x, s_x), \qquad x \in \gamma. \qquad (3.12)$$

3.2 Integration by parts and divergence on the marked Poisson space

Let the marked configuration space Ω_X^M be equipped with the marked Poisson measure $\pi_{\tilde{\sigma}}$. We strengthen the condition (2.4) by demanding that

$$q^{1/2} \in H_0^{1,2}(X \times M). \qquad (3.13)$$

Here, $H_0^{1,2}(X\times M)$ denotes the local Sobolev space of order 1 constructed with respect to the gradient $\nabla^{X\times M}$ in the space $L^2_{\mathrm{loc}}(X;\nu)\otimes L^2(M;\lambda)$, i.e., $H_0^{1,2}(X\times M)$ consists of functions f defined on $X\times M$ such that, for any set $A\in\mathcal{B}_{\mathrm{c}}(X\times M)$, the restriction of f to A coincides with the restriction to A of some function φ from the Sobolev space $H^{1,2}(X\times M)$ constructed as the closure of $\mathfrak{D}$ with respect to the norm

$$\|\varphi\|_{1,2}^2 := \int_{X\times M}\Big(|\nabla^X\varphi(x,m)|^2_{T_x(X)} + |\widetilde\nabla^M\varphi(x,m)|^2_{\mathfrak{g}} + |\varphi(x,s)|^2\Big)\nu(dx)\,\lambda(dm).$$

Additionally, we will suppose that, for each $\Lambda\in\mathcal{B}_{\mathrm{c}}(X)$,

$$|\nabla^G p^\lambda(e,\cdot)|_{\mathfrak{g}}\in L^1(\Lambda_{\mathrm{mk}},\widetilde\sigma), \tag{3.14}$$

where, as before,

$$p^\lambda(g,m)=\frac{dg^*\lambda}{d\lambda}(m).$$

The set $\mathcal{F}C_{\mathrm{b}}^\infty\big(\mathfrak{D},\Omega_X^M\big)$ is a dense subset in the space

$$L^2(\Omega_X^M,\mathcal{B}(\Omega_X^M),\pi_{\widetilde\sigma}) =: L^2(\pi_{\widetilde\sigma}).$$

For any $(v,u)\in\mathfrak{a}$, we have a differential operator in $L^2(\pi_{\widetilde\sigma})$ on the domain $\mathcal{F}C_{\mathrm{b}}^\infty\big(\mathfrak{D},\Omega_X^M\big)$ given by

$$\mathcal{F}C_{\mathrm{b}}^\infty\big(\mathfrak{D},\Omega_X^M\big)\ni F\mapsto\nabla^\Omega_{(v,u)}F\in L^2(\pi_{\widetilde\sigma}).$$

Our aim now is to compute the adjoint operator $\nabla^{\Omega\,*}_{(v,u)}$ in $L^2(\pi_{\widetilde\sigma})$. This corresponds, of course, to the deriving of an integration by parts formula with respect to the measure $\pi_{\widetilde\sigma}$.

But first we present the corresponding formula on $X\times M$.

Definition 3.3 For any $(v,u)\in\mathfrak{a}$, the logarithmic derivative of the measure $\widetilde\sigma$ along (v,u) is defined as the following function on $X\times M$:

$$\beta^{\widetilde\sigma}_{(v,u)} := \beta^{\widetilde\sigma}_v+\beta^{\widetilde\sigma}_u$$

with

$$\beta^{\widetilde\sigma}_v(x,m)=\left\langle\frac{\nabla^X q(x,m)}{q(x,m)},v(x)\right\rangle_{T_x(X)}+\operatorname{div}^X v(x),$$

$\operatorname{div}^X=\operatorname{div}^X_\nu$ being the divergence on X w.r.t. ν, and

$$\beta^{\widetilde\sigma}_u(x,m)=\left\langle\frac{\widetilde\nabla^M q(x,m)}{q(x,m)},u(x)\right\rangle_{\mathfrak{g}}+\langle\nabla^G p^\lambda(e,m),-u(x)\rangle_{\mathfrak{g}}.$$

Upon (3.13), we conclude that, for each $(v,u)\in\mathfrak{a}$, the function $\nabla^{X\times M}_{(v,u)}\log q$ is quadratically integrable with respect to the measure $\widetilde\sigma$, and therefore, since the support of $\nabla^{X\times M}_{(v,u)}\log q$ belongs to $\mathcal{B}_{\mathrm{c}}(X\times M)$, this function is from $L^1(X\times M,\widetilde\sigma)$. Thus, in virtue of the condition (3.14), we get the inclusion $\beta^{\widetilde\sigma}_{(v,u)}\in L^1(X\times M,\widetilde\sigma)$.

By using standard arguments, one shows the following

Lemma 3.1 (Integration by parts formula on $X \times M$) *For all φ_1, $\varphi_2 \in \mathfrak{D}$, we have*

$$\int_{X\times M} (\nabla^{X\times M}_{(v,u)}\varphi_1)(x,m)\varphi_2(x,m)\,\widetilde{\sigma}(dx,dm) =$$
$$= -\int_{X\times M} \varphi_1(x,m)(\nabla^{X\times M}_{(v,u)}\varphi_2)(x,m)\,\widetilde{\sigma}(dx,dm)$$
$$- \int_{X\times M} \varphi_1(x,s)\varphi_2(x,s)\beta^{\widetilde{\sigma}}_{(v,u)}(x,m)\,\widetilde{\sigma}(dx,dm).$$

Remark 3.2 The function $\langle \nabla^G p^\lambda(e,m), -u(x)\rangle_{\mathfrak{g}}$, which appears in the definition of $\beta^{\widetilde{\sigma}}_u$ is, for each fixed $x \in X$, the divergence on M with respect to the measure λ of the vector field $Ru(x)$ on M defined by (3.7), see Remark 3.1. Indeed, for any $u \in \mathfrak{g}$ and for an arbitrary f from $C_0^\infty(M)$—the space of all C^∞ functions on M with compact support, we have

$$\begin{aligned}\int_M \widetilde{\nabla}^M_u f(m)\,\lambda(dm) &= \int_M \langle \nabla^M f(m), (Ru)(m)\rangle_{T_m(M)}\,\lambda(dm)\\ &= \int_M \frac{d}{dt} f(\theta(\exp(tu),m))\big|_{t=0}\,\lambda(dm)\\ &= \int_M f(m)\,\frac{d}{dt} p^\lambda(\exp(tu),m)\big|_{t=0}\,\lambda(dm)\\ &= \int_M f(m)\langle \nabla^G p^\lambda(e,m), u\rangle_{\mathfrak{g}}\,\lambda(dm).\end{aligned}$$

Definition 3.4 For any $(v,u) \in \mathfrak{a}$, the logarithmic derivative of the marked Poisson measure $\pi_{\widetilde{\sigma}}$ along (v,u) is defined as the following function on Ω^M_X:

$$\Omega^M_X \ni \omega \mapsto B^{\pi_{\widetilde{\sigma}}}_{(v,u)}(\omega) := \langle \beta^{\widetilde{\sigma}}_{(v,u)}, \omega\rangle. \tag{3.15}$$

A motivation for this definition is given by the following theorem.

Theorem 3.2 (Integration by parts formula) *For all $F_1, F_2 \in \mathcal{F}C^\infty_{\mathrm{b}}\big(\mathfrak{D},\Omega^M_X\big)$ and each $(v,u) \in \mathfrak{a}$, we have*

$$\begin{aligned}\int_{\Omega^M_X} (\nabla^\Omega_{(v,u)}F_1)(\omega)F_2(\omega)\,\pi_{\widetilde{\sigma}}(d\omega) = &-\int_{\Omega^M_X} F_1(\omega)(\nabla^\Omega_{(v,u)}F_2)(\omega)\,\pi_{\widetilde{\sigma}}(d\omega)\\ &- \int_{\Omega^M_X} F_1(\omega)F_2(\omega)B^{\pi_{\widetilde{\sigma}}}_{(v,u)}(\omega)\,\pi_{\widetilde{\sigma}}(d\omega),\end{aligned} \tag{3.16}$$

or

$$\nabla^{\Omega\,*}_{(v,u)} = -\nabla^\Omega_{(v,u)} - B^{\pi_{\widetilde{\sigma}}}_{(v,u)}(\omega) \tag{3.17}$$

as an operator equality on the domain $\mathcal{F}C^\infty_{\mathrm{b}}\big(\mathfrak{D},\Omega^M_X\big)$ in $L^2(\pi_{\widetilde{\sigma}})$.

Proof. Because of (3.10), the formula (3.17) will be proved if we prove it first for the operator ∇^Ω_v, i.e., when $u(x) \equiv 0$, and then for the operator ∇^Ω_u, i.e., when $v(x) = 0 \in$

$T_x(X)$ for all $x \in X$. We present below only the proof for ∇_u^Ω, since the proof for ∇_v^Ω is basically the same as that of the integration by parts formula in case of Poisson measures [7].

By Proposition 2.2, we have for all $u \in C_0^\infty(X; \mathfrak{g})$

$$\int_{\Omega_X^M} F_1(\eta_t^u(\omega)) F_2(\omega)\, \pi_{\widetilde{\sigma}}(d\omega) = \int_{\Omega_X^M} F_1(\omega) F_2(\eta_{-t}^u(\omega))\, \pi_{\eta_t^u * \widetilde{\sigma}}(d\omega).$$

Differentiating this equation with respect to t, interchanging d/dt with the integrals and setting $t = 0$, the l.h.s. becomes the l.h.s. of (3.16). To see that the r.h.s. then also coincides with the r.h.s. of (3.16), we note that

$$\frac{d}{dt} F_2(\eta_{-t}^u(\omega))\big|_{t=0} = -(\nabla_u^\Omega F_2)(\omega),$$

and by Proposition 2.3

$$\begin{aligned} \frac{d}{dt}\left[\frac{d\pi_{\eta_t^u * \widetilde{\sigma}}}{d\pi_{\widetilde{\sigma}}}(\omega)\right]\Big|_{t=0} &= \sum_{(x,m)\in\omega} \frac{d}{dt}\, p_{\eta_t^u}^{\widetilde{\sigma}}(x,m)\bigg|_{t=0} \\ &= -\langle \beta_u^{\widetilde{\sigma}}, \omega\rangle = -B_u^{\pi_{\widetilde{\sigma}}}(\omega). \quad \blacksquare \end{aligned}$$

Definition 3.5 For a vector field

$$V\colon \Omega_X^M \ni \omega \mapsto V_\omega \in T_\omega(\Omega_X^M),$$

the divergence $\operatorname{div}_{\pi_{\widetilde{\sigma}}}^\Omega V$ is defined via the duality relation

$$\int_{\Omega_X^M} \langle V_\omega, \nabla^\Omega F(\omega)\rangle_{T_\omega\left(\Omega_X^M\right)} \pi_{\widetilde{\sigma}}(d\omega) = -\int_{\Omega_X^M} F(\omega)(\operatorname{div}_{\pi_{\widetilde{\sigma}}}^\Omega V)(\omega)\, \pi_{\widetilde{\sigma}}(d\omega)$$

for all $F \in \mathcal{F}C_{\mathrm{b}}^\infty\big(\mathfrak{D}, \Omega_X^M\big)$, provided it exists (i.e., provided

$$F \mapsto \int_{\Omega_X^M} \langle V_\omega, \nabla^\Omega F(\omega)\rangle_{T_\omega\left(\Omega_X^M\right)} \pi_{\widetilde{\sigma}}(d\omega)$$

is continuous on $L^2(\pi_{\widetilde{\sigma}})$).

A class of smooth vector fields on Ω_X^M for which the divergence can be computed in an explicit form is described in the following proposition.

Proposition 3.1 *For any vector field*

$$V_\omega(x) = \sum_{j=1}^N F_j(\omega)(v_j(x), u_j(x)), \qquad \omega \in \Omega_X^M,\ x \in X,$$

with $F_j \in \mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}, \Omega_X^M\big)$, $(v_j, u_j) \in \mathfrak{a}$, $j = 1, \ldots, N$, *we have*

$$\begin{aligned}(\operatorname{div}_{\pi_{\widetilde{\sigma}}}^{\Omega} V)(\omega) &= \sum_{j=1}^{N} \big(\nabla_{(v_j,u_j)}^{\Omega} F_j\big)(\omega) + \sum_{j=1}^{N} B_{(v_j,u_j)}^{\pi_{\widetilde{\sigma}}}(\omega) F_j(\omega) \\ &= \sum_{j=1}^{N} \langle \nabla^{\Omega} F_j(\omega), (v_j, u_j)\rangle_{T_\omega\left(\Omega_X^M\right)} + \sum_{j=1}^{N} \langle \beta_{(v_j,u_j)}^{\widetilde{\sigma}}, \omega\rangle F_j(\omega).\end{aligned}$$

Proof. Due to the linearity of ∇^{Ω}, it is sufficient to consider the case $N = 1$, i.e., $V_\omega(x) = F_1(\omega)(v(x), u(x))$. By Theorem 3.2, we have for all $F_2 \in \mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}, \Omega_X^M\big)$

$$\begin{aligned}&- \int_{\Omega_X^M} \langle V_\omega, \nabla^{\Omega} F_2(\omega)\rangle_{T_\omega\left(\Omega_X^M\right)} \pi_{\widetilde{\sigma}}(d\omega) = - \int_{\Omega_X^M} F_1(\omega) \nabla_{(v,u)}^{\Omega} F_2(\omega)\, \pi_{\widetilde{\sigma}}(d\omega) \\ &= \int_{\Omega_X^M} \big(\nabla_{(v,u)}^{\Omega} F_1\big)(\omega) F_2(\omega)\, \pi_{\widetilde{\sigma}}(d\omega) + \int_{\Omega_X^M} F_1(\omega) F_2(\omega) B_{(v,u)}^{\pi_{\widetilde{\sigma}}}(\omega)\, \pi_{\widetilde{\sigma}}(d\omega),\end{aligned}$$

which yields

$$\begin{aligned}(\operatorname{div}_{\pi_{\widetilde{\sigma}}}^{\Omega} V)(\omega) &= \nabla_{(v,u)}^{\Omega} F_1(\omega) + B_{(v,u)}^{\pi_{\widetilde{\sigma}}}(\omega) F_1(\omega) \\ &= \langle \nabla^{\Omega} F_1(\omega), (v, u)\rangle_{T_\omega\left(\Omega_X^M\right)} + \langle \beta_{(v,u)}^{\widetilde{\sigma}}, \omega\rangle F_1(\omega). \qquad \blacksquare\end{aligned}$$

Remark 3.3 Extending the definition of $B^{\pi_{\widetilde{\sigma}}}$ in (3.15) to the class of vector fields $V = \sum_{j=1}^{N} F_j \otimes (v_j, u_j)$ by

$$B_V^{\pi_{\widetilde{\sigma}}}(\omega) := \sum_{j=1}^{N} \langle \beta_{(v_j,u_j)}^{\widetilde{\sigma}}, \omega\rangle F_j(\omega) + \sum_{j=1}^{N} \big(\nabla_{(v_j,u_j)}^{\Omega} F_j\big)(\omega),$$

we obtain that

$$\operatorname{div}_{\pi_\sigma}^{\Omega} \bullet = B_{\bullet}^{\pi_{\widetilde{\sigma}}}.$$

In particular, if $(v, u) \in \mathfrak{a}$, it follows, for the "constant" vector field $V_\omega \equiv (v, u)$ on Ω_X^M, that

$$\operatorname{div}_{\pi_{\widetilde{\sigma}}}^{\Omega} (v, u)(\omega) = \langle \operatorname{div}_{\widetilde{\sigma}}^{X\times M} (v, u), \omega\rangle,$$

where $\operatorname{div}_{\widetilde{\sigma}}^{X\times M}(v, u) = \beta_{(v,u)}^{\widetilde{\sigma}}$ is the divergence on $X \times M$ of (v, u) w.r.t. $\widetilde{\sigma}$:

$$\begin{aligned}&\int_{X\times M} \langle \nabla^{X\times M} \varphi(x, m), (v(x), u(x))\rangle_{T_{(x,m)}(X\times M)}\, \widetilde{\sigma}(dx, dm) \\ &\quad = - \int_{X\times M} \varphi(x, m) \big(\operatorname{div}_{\widetilde{\sigma}}^{X\times M}(v, u)\big)(x, m)\, \widetilde{\sigma}(dx, dm), \qquad \varphi \in \mathfrak{D}.\end{aligned}$$

3.3 Integration by parts characterization

In the works [7, 8] it was shown that the mixed Poisson measures are exactly the "volume elements" corresponding to the differential geometry on the configuration space Γ_X. Now, we wish to prove that an analogous statement holds true in our case of Ω_X^M for mixed marked Poisson measures.

We start with a lemma that describes $\widetilde{\sigma}$ as the unique (up to a constant) measure on $X \times M$ with respect to which the divergence $\operatorname{div}_{\widetilde{\sigma}}^{X\times M}$ is the dual operator of the gradient $\nabla^{X\times M}$.

Lemma 3.2 *Let the conditions* (3.13) *and* (3.14) *hold. Then, for every* $\Lambda \in \mathcal{O}_c(X)$ *the measures* $z\widetilde{\sigma}$, $z > 0$, *are the only positive Radon measures* ξ *on* Λ_{mk} *such that* $\operatorname{div}_{\widetilde{\sigma}}^{X\times M}$ *is the dual operator on* $L^2(\Lambda_{\mathrm{mk}};\xi)$ *of* $\nabla^{X\times M}$ *when considered with the domains* $V_0(\Lambda) \times C_0^\infty(\Lambda;\mathfrak{g})$, *resp.* $C_{0,\mathrm{b}}^\infty(\Lambda_{\mathrm{mk}})$ *(i.e., the set of all* $(v,u) \in \mathfrak{a}$, *resp.* $\varphi \in \mathfrak{D}$ *with support in* Λ, *resp.* Λ_{mk}*).*

Proof. In virtue of the conditions (3.13) and (3.14), the lemma is obtained in complete analogy with Remark 4.1 (iii) in [8]. Indeed, let $q_1(x,m)$ and $q_2(x,m)$ be two densities w.r.t. $\nu \otimes \lambda$ for which the logarithmic derivatives coincide. Then, we get

$$\begin{aligned} \nabla_v^X \log q_1(x,m) &= \nabla_v^X \log q_2(x,m), \qquad v \in V_0(X), \\ \widetilde{\nabla}_u^M \log q_1(x,m) &= \widetilde{\nabla}_u^M \log q_2(x,m), \qquad u \in C_0^\infty(\Lambda;\mathfrak{g}),\ \nu\otimes\lambda\text{-a.s.}, \end{aligned}$$

which yields respectively

$$\begin{aligned} q_1(x,m) &= q_2(x,m)c(m), \\ q_1(x,m) &= q_2(x,m)\widetilde{c}(x) \quad \nu\otimes\lambda\text{-a.s.} \end{aligned}$$

Therefore, $q_1(x,m) = \mathrm{const}\, q_2(x,m)$ $\nu\otimes\lambda$-a.s. ■

Let $\varkappa$ be a probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Then, we define a mixed marked Poisson measure as follows:

$$\mu_{\varkappa,\widetilde{\sigma}} = \int_{\mathbb{R}_+} \pi_{z\widetilde{\sigma}}\, \varkappa(dz). \tag{3.18}$$

Here, $\pi_{0\widetilde{\sigma}}$ denotes the Dirac measure on Ω_X^M with mass in $\omega = \{\varnothing\}$. Let $\mathcal{M}_l(\Omega_X^M)$, $l \in [1,\infty)$, denote the set of all probability measures on $(\Omega_X^M, \mathcal{B}(\Omega_X^M))$ such that

$$\int_{\Omega_X^M} |\langle f,\omega\rangle|^l\, \mu(d\omega) < \infty \quad \text{for all } f \in C_{0,\mathrm{b}}(X\times M),\ f \ge 0.$$

Clearly, $\mu_{\varkappa,\widetilde{\sigma}} \in \mathcal{M}_l(\Omega_X^M)$ if and only if

$$\int_{\mathbb{R}_+} z^l\, \varkappa(dz) < \infty. \tag{3.19}$$

We define $(\mathrm{IbP})^{\widetilde{\sigma}}$ to be the set of all $\mu \in \mathcal{M}_1(\Omega_X^M)$ with the property that $\omega \mapsto \langle \beta_{(v,u)}^{\widetilde{\sigma}}, \omega\rangle$ is μ-integrable for all $(v,u) \in \mathfrak{a}$ and which satisfy (3.16) with μ replacing $\pi_{\widetilde{\sigma}}$ for all

$F_1, F_2 \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathfrak{D}, \Omega_X^M)$, $(v, a) \in \mathfrak{g}$. We note that (3.16) makes sense only for such measures and that $B_{(v,u)}^{\pi_{\widetilde{\sigma}}}$ depends only on $\widetilde{\sigma}$ not on $\pi_{\widetilde{\sigma}}$. Obviously, since $\nabla_{(v,u)}^{X\times M}$ obeys the product rule for all $(v, u) \in \mathfrak{a}$, we can always take $F_2 \equiv 1$. Furthermore, $(\mathrm{IbP})^{\widetilde{\sigma}}$ is convex.

Theorem 3.3 *Let the condition* (3.13) *and* (3.14) *be satisfied. Then, the following conditions are equivalent*:
(i) $\mu \in (\mathrm{IbP})^{\widetilde{\sigma}}$;
(ii) $\mu = \mu_{\varkappa,\widetilde{\sigma}}$ *for some probability measure* $\varkappa$ *on* $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ *satisfying* (3.19) *with* $l = 1$.

Proof. The part (ii)⇒(i) is trivial. The proof of (i)⇒(ii) goes along absolutely analogously to that in the particular case where $G = M = \mathbb{R}_+$, see [24]. ■

As a direct consequence of Theorem 3.3, we obtain

Corollary 3.1 *The extreme points of* $(\mathrm{IbP})^{\widetilde{\sigma}}$ *are exactly* $\pi_{z\widetilde{\sigma}}$, $z \geq 0$.

3.4 A lifting of the geometry

Just as in the case of the geometry on the configuration space, we can present an interpretation of the formulas obtained in subsections 3.1–3.3 via a simple "lifting rule."

Suppose that $f \in C_{0,\mathrm{b}}(X \times M)$, or more generally f is an arbitrary measurable function on $X \times M$ for which there exists (depending on f) $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$ such that $\operatorname{supp} f \subset \Lambda_{\mathrm{mk}}$. Then, f generates a (cylinder) function on Ω_X^M by the formula

$$L_f(\omega) := \langle f, \omega \rangle, \qquad \omega \in \Omega_X^M.$$

We will call L_f the lifting of f.

As before, any vector field $(v, u) \in \mathfrak{a}$,

$$(v, u)\colon X \ni x \mapsto (v(x), u(x)) \in T_{(x,m)}(X \times M) = T_x(X) \dotplus \mathfrak{g},$$

can be considered as a vector field on Ω_X^M (the lifting of (v, u)), which we denote by $L_{(v,u)}$:

$$L_{(v,u)}\colon \Omega_X^M \ni \omega = \{\gamma, s\} \mapsto \{x \mapsto (v(x), u(x))\} \in T_\omega(\Omega_X^M) = L^2(X \to T(X) \dotplus \mathfrak{g}\,; \gamma).$$

For $(v_1, u_1), (v_2, u_2) \in \mathfrak{a}$, the formula (3.11) can be written as follows:

$$\big\langle L_{(v_1,u_1)}, L_{(v_2,u_2)} \big\rangle_{T_\omega(\Omega_X^M)} = L_{\langle (v_1,u_1),(v_2,u_2)\rangle_{T(X\times M)}}(\omega),$$

i.e., the scalar product of lifted vector fields is computed as the lifting of the scalar product

$$\langle (v_1(x), u_2(x)), (v_2(x), u_2(x)) \rangle_{T_{(x,s_x)}(X\times M)} = f(x).$$

This rule can be used as a definition of the tangent space $T_\omega(\Omega_X^M)$.

The formula (3.9) has now the following interpretation:

$$\big(\nabla_{(v,u)}^{\Omega} L_\varphi\big)(\omega) = L_{\nabla_{(v,u)}^{X\times M}\varphi}(\omega), \qquad \varphi \in \mathfrak{D},\ \omega \in \Omega_X^M, \tag{3.20}$$

and the "lifting rule" for the gradient is given by

$$(\nabla^{\Omega} L_{\varphi})(\gamma, s) \colon \gamma \ni x \mapsto \nabla^{X\times M} \varphi(x, s_x). \tag{3.21}$$

As follows from (3.15), the logarithmic derivative $B^{\pi_{\widetilde{\sigma}}}_{(v,u)} \colon \Omega^M_X \to \mathbb{R}$ is obtained via the lifting procedure of the corresponding logarithmic derivative $\beta^{\widetilde{\sigma}}_{(v,u)} \colon X \times M \to \mathbb{R}$, namely,

$$B^{\pi_{\widetilde{\sigma}}}_{(v,u)}(\omega) = L_{\beta^{\widetilde{\sigma}}_{(v,u)}}(\omega),$$

or equivalently, one has for the divergence of a lifted vector field:

$$\operatorname{div}^{\Omega}_{\pi_{\widetilde{\sigma}}} L_{(v,a)} = L_{\operatorname{div}^{X\times M}_{\widetilde{\sigma}}}(v, a). \tag{3.22}$$

We underline that by (3.20) and (3.21) one recovers the action of $\nabla^{\Omega}_{(v,a)}$ and ∇^{Ω} on all functions from $\mathcal{F}C^{\infty}_{\mathrm{b}}\big(\mathfrak{D}, \Omega^M_X\big)$ algebraically from requiring the product or the chain rule to hold. Also, the action of $\operatorname{div}^{\Omega}_{\pi_{\widetilde{\sigma}}}$ on more general cylindrical vector fields follows as in Remark 3.3 if one assumes the usual product rule for $\operatorname{div}_{\pi^{\Omega}_{\widetilde{\sigma}}}$ to hold.

4 Representations of the Lie algebra $\mathfrak{a}$ of the group $\mathfrak{A}$

Using the $\mathfrak{A}$-quasiinvariance of $\pi_{\widetilde{\sigma}}$, we can define the unitary representation of the group $\mathfrak{A} = \operatorname{Diff}_0(X) \underset{\alpha}{\times} G^X$ in the space $L^2(\pi_{\widetilde{\sigma}})$. Namely, for $a \in \mathfrak{A}$, we define the unitary operator

$$\big(V_{\pi_{\widetilde{\sigma}}}(a)F\big)(\omega) := F(a(\omega)) \sqrt{\frac{da^{-1*}\pi_{\widetilde{\sigma}}}{d\pi_{\widetilde{\sigma}}}(\omega)}, \qquad F \in L^2(\pi_{\widetilde{\sigma}}).$$

Then, we have

$$V_{\pi_{\widetilde{\sigma}}}(a_1)V_{\pi_{\widetilde{\sigma}}}(a_2) = V_{\pi_{\widetilde{\sigma}}}(a_1 a_2), \qquad a_1, a_2 \in \mathfrak{A}.$$

As has been noted in Introduction, this representation is reducible, cf. [24]

As in subsec. 3.1, to any vector field $v \in V_0(X)$ there corresponds a one-parameter subgroup of diffeomorphisms ψ^v_t, $t \in \mathbb{R}$. It generates a one-parameter unitary group

$$V_{\pi_{\widetilde{\sigma}}}(\psi^v_t) := \exp[itJ_{\pi_{\widetilde{\sigma}}}(v)], \qquad t \in \mathbb{R},$$

where $J_{\pi_{\widetilde{\sigma}}}(v)$ denotes the selfadjoint generator of this group. Analogously, to a subgroup η^u_t, $u \in C^{\infty}_0(X; \mathfrak{g})$, there corresponds a one-parameter unitary group

$$V_{\pi_{\widetilde{\sigma}}}(\eta^u_t) := \exp[itI_{\pi_{\widetilde{\sigma}}}(u)]$$

with a generator $I_{\pi_{\widetilde{\sigma}}}(u)$.

Proposition 4.1 *For any $v \in V_0(X)$ and $u \in C^{\infty}_0(X; \mathfrak{g})$, the following operator equalities on the domain $\mathcal{F}C^{\infty}_{\mathrm{b}}\big(\mathfrak{D}, \Omega^M_X\big)$ hold:*

$$J_{\pi_{\widetilde{\sigma}}}(v) = \frac{1}{i}\, \nabla^{\Omega}_v + \frac{1}{2i}\, B^{\pi_{\widetilde{\sigma}}}_v,$$
$$I_{\pi_{\widetilde{\sigma}}}(u) = \frac{1}{i}\, \nabla^{\Omega}_u + \frac{1}{2i}\, B^{\pi_{\widetilde{\sigma}}}_u.$$

Proof. These equalities follow immediately from the definition of the directional derivatives ∇_v^{Ω} and ∇_a^{Ω}, Theorem 3.2, and the form of the operators $V_{\pi_{\tilde{\sigma}}}(\psi_t^v)$ and $V_{\pi_{\tilde{\sigma}}}(\theta_t^u)$. ■

For any $(v,u) \in \mathfrak{a}$, define an operator

$$\mathcal{R}_{\pi_{\tilde{\sigma}}}(v,u) := J_{\pi_{\tilde{\sigma}}}(v) + I_{\pi_{\tilde{\sigma}}}(u).$$

By Proposition 4.1,

$$\mathcal{R}_{\pi_{\tilde{\sigma}}}(v,u) = \frac{1}{i}\nabla^{\Omega}_{(v,u)} + \frac{1}{2i}B^{\pi_{\tilde{\sigma}}}_{(v,u)}.$$

We wish to derive now a commutation relation between these operators.

Lemma 4.1 *The Lie-bracket $[(v_1,u_1),(v_2,u_2)]$ of the vector fields (v_1,u_1), $(v_2,u_2) \in \mathfrak{a}$, i.e., a vector field from $\mathfrak{a}$ such that*

$$\nabla^{X\times M}_{[(v_1,u_1),(v_2,u_2)]} = \nabla^{X\times M}_{(v_1,u_1)}\nabla^{X\times M}_{(v_2,u_2)} - \nabla^{X\times M}_{(v_2,u_2)}\nabla^{X\times M}_{(v_1,u_1)} \quad \text{on } \mathfrak{D},$$

is given by

$$[(v_1,u_1),(v_2,u_2)] = ([v_1,v_2], \nabla^X_{v_1}u_2 - \nabla^X_{v_2}u_1 + [u_1,u_2]),$$

where $[v_1,v_2]$ is the Lie-bracket of the vector fields v_1, v_2 on X,

$$[u_1,u_2](x) = [u_1(x),u_2(x)]$$

(the latter being the Lie-bracket on $\mathfrak{g}$ of $u_1(x),u_2(x) \in \mathfrak{g}$), and $\nabla^X_v u$ is the derivative in direction v of a $\mathfrak{g}$-valued function u on X.

Proof. First, we have on $\mathfrak{D}$:

$$\nabla^X_{v_1}\nabla^X_{v_2} - \nabla^X_{v_2}\nabla^X_{v_1} = \nabla^X_{[v_1,v_2]}, \qquad v_1,v_2 \in V_0(X). \tag{4.1}$$

Next, using (3.5),

$$\widetilde{\nabla}^M_u f(x,m) = \langle \nabla^G \hat{f}(x,e,m), u(x)\rangle_{\mathfrak{g}}, \qquad \hat{f}(x,g,m) := f(x,\theta(g,m)),$$

and so

$$\begin{aligned}
&(\widetilde{\nabla}^M_{u_1}\widetilde{\nabla}^M_{u_2} - \widetilde{\nabla}^M_{u_2}\widetilde{\nabla}^M_{u_1})f(x,m)\\
&\quad = \langle \nabla^G \hat{f}(x,e,m), [u_1(x),u_2(x)]\rangle_{\mathfrak{g}}\\
&\quad = \widetilde{\nabla}^M_{[u_1,u_2]}f(x,m), \qquad u_1,u_2 \in C_0^\infty(X;\mathfrak{g}).
\end{aligned} \tag{4.2}$$

Finally,

$$\begin{aligned}
&(\nabla^X_v\widetilde{\nabla}^M_u - \widetilde{\nabla}^M_u\nabla^X_v)f(x,m)\\
&= \langle \nabla^X\langle \nabla^G \hat{f}(x,e,m), u(x)\rangle_{\mathfrak{g}}, v(x)\rangle_{T_x(X)}\\
&\quad - \langle \nabla^G\langle \nabla^X \hat{f}(x,e,m), v(x)\rangle_{T_x(X)}, u(x)\rangle_{\mathfrak{g}}\\
&= \langle \nabla^X\nabla^G \hat{f}(x,e,m), v(x)\otimes u(x)\rangle_{T_x(X)\otimes\mathfrak{g}} + \langle \nabla^G \hat{f}(x,e,m), \nabla^X_v u(x)\rangle_{\mathfrak{g}}\\
&\quad - \langle \nabla^G\nabla^X \hat{f}(x,e,m), u(x)\otimes v(x)\rangle_{\mathfrak{g}\otimes T_x(X)}\\
&= \langle \nabla^G \hat{f}(x,e,m), \nabla^X_v u(x)\rangle_{\mathfrak{g}} = \widetilde{\nabla}^M_{\nabla^X_v u}f(x,m),\\
&\qquad v \in V_0(X),\ u \in C_0^\infty(X;\mathfrak{g}).
\end{aligned} \tag{4.3}$$

The equalities (4.1)–(4.3) yield the lemma. ■

Proposition 4.2 *For arbitrary* (v_1, u_1), $(v_2, u_2) \in \mathfrak{a}$, *the following operator equality holds on* $\mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}, \Omega_X^M\big)$:

$$[\mathcal{R}_{\pi_{\widetilde{\sigma}}}(v_1, u_1), \mathcal{R}_{\pi_{\widetilde{\sigma}}}(v_2, u_2)] = \mathcal{R}_{\pi_{\widetilde{\sigma}}}([(v_1, u_1), (v_2, u_2)]).$$

In particular,

$$\begin{aligned} [J_{\pi_{\widetilde{\sigma}}}(v_1), J_{\pi_{\widetilde{\sigma}}}(v_2)] &= -iJ_{\pi_{\widetilde{\sigma}}}([v_1, v_2]), \qquad && v_1, v_2 \in V_0(X), \\ [I_{\pi_{\widetilde{\sigma}}}(u_1), I_{\pi_{\widetilde{\sigma}}}(u_2)] &= -I_{\pi_{\widetilde{\sigma}}}([u_1, u_2]), && u_1, u_2 \in C_0^{\infty}(X; \mathfrak{g}), \\ [J_{\pi_{\widetilde{\sigma}}}(v), I_{\pi_{\widetilde{\sigma}}}(u)] &= -iI_{\pi_{\widetilde{\sigma}}}(\nabla_v^X u), && v \in V_0(X),\ u \in C_0^{\infty}(X; \mathfrak{g}). \end{aligned}$$

Proof. First we note that Lemma 4.1 and (3.9) immediately imply

$$\nabla^{\Omega}_{(v_1,u_1)}\nabla^{\Omega}_{(v_2,u_2)} - \nabla^{\Omega}_{(v_2,u_2)}\nabla^{\Omega}_{(v_1,u_1)} = \nabla^{\Omega}_{[(v_1,u_1),(v_2,u_2)]} \quad \text{on } \mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}, \Omega_X^M\big).$$

Therefore, by using the chain rule, we conclude that the lemma will be proved if we show that

$$\nabla^{\Omega}_{(v_1,u_1)} B^{\pi_{\widetilde{\sigma}}}_{(v_2,u_2)} - \nabla^{\Omega}_{(v_2,u_2)} B^{\pi_{\widetilde{\sigma}}}_{(v_1,u_1)} = B^{\pi_{\widetilde{\sigma}}}_{[(v_1,u_1),(v_2,u_2)]} \quad \pi_{\widetilde{\sigma}}\text{-a.e.} \tag{4.4}$$

But upon the representation

$$B^{\pi_{\widetilde{\sigma}}}_{(v,u)}(\omega) = \langle \nabla_v^X \log q + \widetilde{\nabla}_u^M \log q + \operatorname{div}^X v + \langle \nabla^G p^{\lambda}(e, m), -u(x)\rangle_{\mathfrak{g}}, \omega\rangle$$

and Remark 3.2, we easily derive (4.4) again from Lemma 4.1. ■

Thus, the operators $\mathcal{R}_{\pi_{\widetilde{\sigma}}}(v, u)$, $(v, u) \in \mathfrak{a}$, give a marked Poisson space representation of the Lie algebra $\mathfrak{a}$ of the group $\mathfrak{A}$.

5 Intrinsic Dirichlet forms on marked Poisson spaces

5.1 Definition of the intrinsic Dirichlet form

From now on, the underlying space of "nice functions" on $X \times M$ will be instead of $\mathfrak{D}$ the space $\mathfrak{D}_0 := C_0^{\infty}(X \times M)$ consisting of all C^{∞} functions with compact support in $X \times M$. Evidently, $\mathfrak{D}_0$ is a subset of $\mathfrak{D}$ and in the case where M is itself compact $\mathfrak{D}_0 = \mathfrak{D}$. Absolutely analogously to $\mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}, \Omega_X^M\big)$ one constructs the set $\mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}_0, \Omega_X^M\big)(\subset \mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}, \Omega_X^M\big))$, which is dense in $L^2(\pi_{\widetilde{\sigma}})$. By $\mathcal{F}\mathcal{P}(\mathfrak{D}_0, \Omega_X^M)$ we denote the set of all cylinder functions of the form (3.8) in which the functions $\varphi_1, \dots, \varphi_N$ belong to $\mathfrak{D}_0$ and the generating function g_F is a polynomial on $\mathbb{R}^N$, i.e., $g_F \in \mathcal{P}(\mathbb{R}^N)$. Finally, in the same way we introduce $\mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0, \Omega_X^M)$ where $g_F \in C_{\mathrm{p}}^{\infty}(\mathbb{R}^N)$ (:=the set of all C^{∞}-functions f on $\mathbb{R}^N$ such that f and its partial derivatives of any order are polynomially bounded).

We have obviously

$$\begin{aligned} \mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}_0, \Omega_X^M\big) &\subset \mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0, \Omega_X^M), \\ \mathcal{F}\mathcal{P}(\mathfrak{D}_0, \Omega_X^M) &\subset \mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0, \Omega_X^M), \end{aligned}$$

and these are algebras with respect to the usual operations. The existence of the Laplace transform $\ell_{\pi_{\widetilde{\sigma}}}(f)$ for each $f \in C_0(X \times M)$ implies, in particular, that $\mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0, \Omega_X^M) \subset L^2(\pi_{\widetilde{\sigma}})$.

Definition 5.1 For $F_1, F_2 \in \mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0, \Omega_X^M)$, we introduce a pre-Dirichlet form as

$$\mathcal{E}_{\pi_{\widetilde{\sigma}}}^{\Omega}(F_1, F_2) = \int_{\Omega_X^M} \langle \nabla^{\Omega} F_1(\omega), \nabla^{\Omega} F_2(\omega)\rangle_{T_\omega(\Omega_X^M)} \, \pi_{\widetilde{\sigma}}(d\omega). \tag{5.1}$$

Note that, for all $F \in \mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0, \Omega_X^M)$, the formula (3.12) is still valid and therefore, for $F_1 = g_{F_1}(\langle \varphi_1, \cdot\rangle, \dots, \langle \varphi_N, \cdot\rangle)$ and $F_2 = g_{F_2}(\langle \xi_1, \cdot\rangle, \dots, \langle \xi_K, \cdot\rangle)$ from $\mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0, \Omega_X^M)$, we have

$$\begin{aligned} &\langle \nabla^{\Omega} F_1(\omega), \nabla^{\Omega} F_2(\omega)\rangle_{T_\omega(\Omega_X^M)} = \\ &= \sum_{j=1}^{N}\sum_{k=1}^{K} \frac{\partial g_{F_1}}{\partial r_j}(\langle \varphi_1, \omega\rangle, \dots, \langle \varphi_N, \omega\rangle) \frac{\partial g_{F_2}}{\partial r_k}(\langle \xi_1, \omega\rangle, \dots, \langle \xi_K, \omega\rangle)\times \\ &\quad \times \int_X \langle \nabla^{X\times M} \varphi_j(x, s_x), \nabla^{X\times M} \xi_k(x, s_x)\rangle_{T_{(x,s_x)}(X\times M)} \, \gamma(dx) \\ &= \sum_{j=1}^{N}\sum_{k=1}^{K} \frac{\partial g_{F_1}}{\partial r_j}(\langle \varphi_1, \omega\rangle, \dots, \langle \varphi_N, \omega\rangle) \frac{\partial g_{F_2}}{\partial r_k}(\langle \xi_1, \omega\rangle, \dots, \langle \xi_K, \omega\rangle)\times \\ &\qquad \times \langle \langle \nabla^{X\times M} \varphi_j, \nabla^{X\times M} \xi_k\rangle_{T(X\times M)}, \omega\rangle. \end{aligned} \tag{5.2}$$

Since for $\varphi, \xi \in \mathfrak{D}_0$, the function

$$\begin{aligned} &\langle \nabla^{X\times M} \varphi(x, m), \nabla^{X\times M} \xi(x, m)\rangle_{T_{(x,m)}(X\times M)} = \\ &= \langle \nabla^{X} \varphi(x, m), \nabla^{X} \xi(x, m)\rangle_{T_x(X)} + \langle \widetilde{\nabla}^{M} \varphi(x, m), \widetilde{\nabla}^{M} \xi(x, m)\rangle_{\mathfrak{g}} \end{aligned}$$

belongs to $\mathfrak{D}_0$, we conclude that

$$\langle \nabla^{\Omega} F_1(\cdot), \nabla^{\Omega}(\cdot) F_2(\cdot)\rangle_{T(\Omega_X^M)} \in L^1(\pi_{\widetilde{\sigma}}), \qquad F_1, F_2 \in \mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0, \Omega_X^M),$$

and so (5.1) is well defined.

We will call $\mathcal{E}_{\pi_{\widetilde{\sigma}}}^{\Omega}$ the intrinsic pre-Dirichlet form corresponding to the marked Poisson measure $\pi_{\widetilde{\sigma}}$ on Ω_X^M. In the next subsection we will prove the closability of $\mathcal{E}_{\pi_{\widetilde{\sigma}}}^{\Omega}$.

5.2 Intrinsic Dirichlet operators

We start with introducing the pre-Dirichlet operator corresponding to the measure $\widetilde{\sigma}$ on $X \times M$ and to the gradient $\nabla^{X\times M}$:

$$\mathcal{E}_{\widetilde{\sigma}}^{X\times M}(\varphi, \xi) := \int_{X\times M} \langle \nabla^{X\times M} \varphi(x, m), \nabla^{X\times M} \xi(x, m)\rangle_{T_{(x,m)}(X\times M)} \, \widetilde{\sigma}(dx, dm), \tag{5.3}$$

where φ, $\xi \in \mathfrak{D}_0$. This form is associated with the Dirichlet operator

$$H_{\widetilde{\sigma}}^{X\times M} := H_{\widetilde{\sigma}}^{X} + H_{\widetilde{\sigma}}^{M} \tag{5.4}$$

on $\mathfrak{D}_0$ which satisfies

$$\mathcal{E}_{\widetilde{\sigma}}^{X\times M}(\varphi, \xi) = (H_{\widetilde{\sigma}}^{X\times M} \varphi, \xi)_{L^2(\widetilde{\sigma})}, \qquad \varphi,\ \xi \in \mathfrak{D}_0. \tag{5.5}$$

Here, $H^X_{\widetilde\sigma}$ and $H^M_{\widetilde\sigma}$ are the Dirichlet operators of ∇^X and $\widetilde\nabla^M$, respectively. Evidently,

$$H^X_{\widetilde\sigma}\varphi(x,m) = -\Delta^X\varphi(x,m) - \langle\nabla^X\log q(x,m),\nabla^X\varphi(x,m)\rangle_{T_x(X)}, \tag{5.6}$$

where Δ^X denotes the Laplace-Beltrami operator corresponding to ∇^X.

Let us calculate the operator $H^M_{\widetilde\sigma}$. Suppose $f\in\mathfrak{D}_0$ and $W\in C_0(X\times M;\mathfrak{g})$. Analogously to Remark 3.1, we conclude

$$\langle\widetilde\nabla^M f(x,m),W(x,m)\rangle_{\mathfrak{g}} = \langle\nabla^M f(x,m),(RW)(x,m)\rangle_{T_m(M)}, \tag{5.7}$$

where $RW\in C_0^\infty(X\times M;TM)$ is given by

$$X\times M\ni(x,m)\mapsto(RW)(x,m) := \frac{d}{dt}\theta(\exp(tW(x,m)),m)\big|_{t=0}\in T_mM. \tag{5.8}$$

Therefore, using the integration by parts formula on M for a vector field with a compact support, we get

$$\begin{aligned}\int_{X\times M}&\langle\widetilde\nabla^M f(x,m),W(x,m)\rangle_{\mathfrak{g}}\,\widetilde\sigma(dx,dm)\\ &= -\int_{X\times M} f(x,m)\big[\operatorname{div}^M(RW)(x,m)\\ &\quad+\langle\nabla^M\log q(x,m),(RW)(x,m)\rangle_{T_m(M)}\big]\,\widetilde\sigma(dx,dm)\\ &= -\int_{X\times M} f(x,m)\big[\operatorname{div}^M(RW)(x,m)+\langle\widetilde\nabla^M\log q(x,m),W(x,m)\rangle_{\mathfrak{g}}\big]\,\widetilde\sigma(dx,dm),\end{aligned}$$

where div^M is the divergence on M with respect to the usual gradient ∇^M and the measure λ. Thus, the divergence $\widetilde{\operatorname{div}}^M_{\widetilde\sigma}$ on $X\times M$ w.r.t. the gradient $\widetilde\nabla^M$ and the measure $\widetilde\sigma$ is given by

$$\widetilde{\operatorname{div}}^M_{\widetilde\sigma}W(x,m) = \operatorname{div}^M(RW)(x,m)+\langle\widetilde\nabla^M\log q(x,m),W(x,m)\rangle_{\mathfrak{g}}.$$

In particular, the divergence $\widetilde{\operatorname{div}}^M$ w.r.t. the measure $\nu(dx)\,\lambda(dm)$ equals

$$\widetilde{\operatorname{div}}^M W(x,m) = \operatorname{div}^M(RW)(x,m). \tag{5.9}$$

It is easy to see that, for $f\in\mathfrak{D}_0$, $W=\widetilde\nabla^M f\in C_0^\infty(X\times M;\mathfrak{g})$, and so we have finally

$$H^M_{\widetilde\sigma}f = \widetilde{\operatorname{div}}^M\widetilde\nabla^M f = -\widetilde\Delta^M f - \langle\widetilde\nabla^M\log q,\widetilde\nabla^M f\rangle_{\mathfrak{g}}, \qquad f\in\mathfrak{D}_0, \tag{5.10}$$

where

$$\widetilde\Delta^M f = \widetilde{\operatorname{div}}^M\widetilde\nabla^M f := \operatorname{div}^M(R(\widetilde\nabla^M f)). \tag{5.11}$$

The closure of the form $\mathcal{E}^{X\times M}_{\widetilde\sigma}$ on

$$L^2(X\times M;\widetilde\sigma) =: L^2(\widetilde\sigma)$$

is denoted by $(\mathcal{E}_{\widetilde{\sigma}}^{X\times M}, D(\mathcal{E}_{\widetilde{\sigma}}^{X\times M}))$. This form generates a positive selfadjoint operator in $L^2(\widetilde{\sigma})$ (the so-called Friedrichs extension of $H_{\widetilde{\sigma}}^{X\times M}$, see e.g. [9]). For this extension we preserve the notation $H_{\widetilde{\sigma}}^{X\times M}$ and denote the domain by $D(H_{\widetilde{\sigma}}^{X\times M})$.

Let us introduce a differential operator $H_{\pi_{\widetilde{\sigma}}}^{\Omega}$ on the domain $\mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}_0, \Omega_X^M\big)$ which is given on any $F \in \mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}_0, \Omega_X^M\big)$ of the form (3.8) by the formula

$$
\begin{aligned}
(H_{\pi_{\widetilde{\sigma}}}^{\Omega} F)(\omega) := &-\sum_{j,k=1}^{N} \frac{\partial^2 F}{\partial r_j \partial r_k}(\langle\varphi_1,\omega\rangle,\dots,\langle\varphi_n,\omega\rangle)\langle\langle\nabla^{X\times M}\varphi_j, \nabla^{X\times M}\varphi_k\rangle_{T(X\times M)},\omega\rangle \\
&+\sum_{j=1}^{N} \frac{\partial F}{\partial r_j}(\langle\varphi_1,\omega\rangle,\dots,\langle\varphi_n,\omega\rangle)\langle H_{\widetilde{\sigma}}^{X\times M}\varphi_j,\omega\rangle. \qquad (5.12)
\end{aligned}
$$

Since

$$\langle\nabla^{X\times M}\log q, \nabla^{X\times M}\varphi_j\rangle_{T(X\times M)} \in L^2(\widetilde{\sigma}) \cap L^1(\widetilde{\sigma})$$

(see condition (3.13)), the r.h.s. of (5.12) is well defined as an element of $L^2(\pi_{\widetilde{\sigma}})$. The following theorem implies, in particular, that $H_{\pi_{\widetilde{\sigma}}}^{\Omega}$ is well defined as a linear operator on $\mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}_0, \Omega_X^M\big)$, i.e., independently of the representation of F as in (3.8).

Theorem 5.1 *The operator $H_{\pi_{\widetilde{\sigma}}}^{\Omega}$ is associated with the intrinsic Dirichlet form $\mathcal{E}_{\pi_{\widetilde{\sigma}}}^{\Omega}$ in the sense that, for all $F_1, F_2 \in \mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}_0, \Omega_X^M\big)$*

$$\mathcal{E}_{\pi_{\widetilde{\sigma}}}^{\Omega}(F_1,F_2) = (H_{\pi_{\widetilde{\sigma}}}^{\Omega} F_1, F_2)_{L^2(\pi_{\widetilde{\sigma}})}, \qquad (5.13)$$

or

$$H_{\pi_{\widetilde{\sigma}}}^{\Omega} = -\operatorname{div}_{\pi_{\widetilde{\sigma}}}^{\Omega} \nabla^{\Omega} \quad \text{on } \mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}_0, \Omega_X^M\big).$$

We call $H_{\pi_{\widetilde{\sigma}}}^{\Omega}$ the intrinsic Dirichlet operator of the measure $\pi_{\widetilde{\sigma}}$.

Lemma 5.1 *For any $\varphi \in \mathfrak{D}_0$ and $W \in C_0^{\infty}(X\times M;\mathfrak{g})$, we have*

$$\widetilde{\operatorname{div}}^M(\varphi W)(x,m) = \langle\widetilde{\nabla}^M\varphi(x,m), W(x,m)\rangle_{\mathfrak{g}} + \varphi(x,m)\widetilde{\operatorname{div}}^M W(x,m).$$

Proof. By (5.7), (5.8), and (5.9)

$$
\begin{aligned}
\widetilde{\operatorname{div}}^M(\varphi W)(x,m) &= \operatorname{div}^M\Big[\frac{d}{dt}\,\theta(\exp(t\varphi(x,m)W(x,m)),m)\big|_{t=0}\Big] \\
&= \operatorname{div}^M\Big[\varphi(x,m)\,\frac{d}{dt}\,\theta(\exp(tW(x,m)),m)\big|_{t=0}\Big] \\
&= \langle\nabla^M\varphi(x,m), \frac{d}{dt}\,\theta(\exp(tW(x,m)),m)\big|_{t=0}\rangle_{T_m(M)} \\
&\quad + \varphi(x,m)\operatorname{div}^M\Big[\frac{d}{dt}\theta(\exp(tW(x,m)),m)\big|_{t=0}\Big] \\
&= \frac{d}{dt}\,\varphi(x,\theta(\exp(tW(x,m)),m))\big|_{t=0} + \varphi(x,m)\widetilde{\operatorname{div}}^M W(x,m) \\
&= \langle\widetilde{\nabla}^M\varphi(x,m), W(x,m)\rangle_{\mathfrak{g}} + \varphi(x,m)\widetilde{\operatorname{div}}^M W(x,m). \quad \blacksquare
\end{aligned}
$$

Proof of Theorem 5.1. For shortness of notations we will prove the formula (5.13) in the case where $F_1, F_2 \in \mathcal{F}C_{\mathrm{b}}^{\infty}\big(\mathfrak{D}_0, \Omega_X^M\big)$ are of the form

$$F_1 = g_{F_1}(\langle\varphi,\omega\rangle), \quad F_2 = g_{F_2}(\langle\xi,\omega\rangle).$$

However, it is a trivial step to generalize the proof to general F_1, F_2.

Let $\Lambda \in \mathcal{O}_{\mathrm{c}}(X)$ be chosen so that the supports of the functions φ and ξ are in Λ_{mk}. Then, by (5.1), (5.2), and the construction of the marked Poisson measure

$$\begin{aligned}
\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}}(F_1,F_2) &= \int_{\Omega_X^M} g'_{F_1}(\langle\varphi,\omega\rangle) g'_{F_2}(\langle\xi,\omega\rangle)\langle\langle\nabla^{X\times M}\varphi, \nabla^{X\times M}\xi\rangle_{T(X\times M)},\omega\rangle\,\pi_{\widetilde{\sigma}}(d\omega)\\
&= -e^{\widetilde{\sigma}(\Lambda_{\mathrm{mk}})}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda_{\mathrm{mk}}^n} g'_{F_1}(\varphi(x_1,m_1)+\cdots+\varphi(x_n,m_n))\\
&\qquad\times g'_{F_2}(\xi(x_1,m_1)+\cdots+\xi(x_n,m_n))\\
&\times\left[\sum_{i=1}^{n}\langle\nabla^{X\times M}\varphi(x_i,m_i), \nabla^{X\times M}\xi(x_i,m_i)\rangle_{T_{(x_i,m_i)}(X\times M)}\right]\widetilde{\sigma}(dx_1,dm_1)\cdots\widetilde{\sigma}(dx_1,dm_1)\\
&= e^{-\widetilde{\sigma}(\Lambda_{\mathrm{mk}})}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda_{\mathrm{mk}}^n}\sum_{i=1}^{n}\langle\nabla_i^{X\times M} g_{F_1}(\varphi(x_1,m_1)+\cdots+\varphi(x_n,m_n)),\\
&\nabla_i^{X\times M} g_{F_2}(\xi(x_1,m_1)+\cdots+\xi(x_n,m_n))\rangle_{T_{(x_i,m_i)}(X\times M)}\,\widetilde{\sigma}(dx_1,dm_1)\cdots\widetilde{\sigma}(dx_n,dm_n),
\end{aligned}$$

where $\nabla_i^{X\times M}$ denotes the $\nabla^{X\times M}$ gradient in the (x_i,m_i) variables. Therefore, by using (5.10) and Lemma 5.1, we proceed in the calculation of $\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}}(F_1,F_2)$ as follows:

$$\begin{aligned}
&= e^{-\widetilde{\sigma}(\Lambda_{\mathrm{mk}})}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda_{\mathrm{mk}}^n}\left[\sum_{i=1}^{n} H_{\widetilde{\sigma}}^{(X\times M)_i} g_{F_1}(\varphi(x_1,m_1)+\cdots+\varphi(x_n,m_n))\right]\times\\
&\qquad\times g_{F_1}(\xi(x_1,m_1)+\cdots+\xi(x_n,m_n))\,\widetilde{\sigma}(x_1,m_1)\cdots\widetilde{\sigma}(dx_n,dm_n)\\
&= -e^{\widetilde{\sigma}(\Lambda_{\mathrm{mk}})}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda_{\mathrm{mk}}^n}\Bigg[\sum_{i=1}^{n} g''_{F_1}(\varphi(x_1,m_1)+\cdots+\varphi(x_n,m_n))\times\\
&\qquad\times\langle\nabla^{X\times M}\varphi(x_i,m_i), \nabla^{X\times M}\varphi(x_i,m_i)\rangle_{T_{(x_i,m_i)}(X\times M)}\\
&\qquad+ g'_{F_1}(\varphi(x_1,m_1)+\cdots+\varphi(x_n,m_n)) H_{\widetilde{\sigma}}^{X\times M}\varphi(x_i,m_i)\Bigg]\times\\
&\qquad\times g_{F_2}(\xi(x_1,m_1)+\cdots+\xi(x_n,m_n))\,\widetilde{\sigma}(dx_1,dm_1)\cdots\widetilde{\sigma}(dx_n,dm_n)\\
&= \int_{\Omega_X^M} H^{\Omega}_{\pi_{\widetilde{\sigma}}} F_1(\omega)F_2(\omega)\,\pi_{\widetilde{\sigma}}(d\omega). \quad \blacksquare
\end{aligned}$$

Remark 5.1 The operator $H^{\Omega}_{\pi_{\widetilde{\sigma}}}$ can be naturally extended to cylinder functions of the form

$$F(\omega) := e^{\langle\varphi,\omega\rangle}, \qquad \varphi \in \mathfrak{D}_0,\ \omega \in \Omega_X^M,$$

since such F belong to $L^2(\pi_{\widetilde{\sigma}})$. We then have

$$H^{\Omega}_{\pi_{\widetilde{\sigma}}} e^{\langle\varphi,\omega\rangle} = \langle H^{X\times M}_{\widetilde{\sigma}}\varphi - |\nabla^{X\times M}\varphi|^2_{T(X\times M)}, \omega\rangle\, e^{\langle\varphi,\omega\rangle}. \tag{5.14}$$

As an immediate consequence of Theorem 5.1 we obtain

Corollary 5.1 $(\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}}, \mathcal{F}C^{\infty}_{\mathrm{b}}(\mathfrak{D}_0, \Omega^M_X))$ *is closable on* $L^2(\pi_{\widetilde{\sigma}})$*. Its closure* $(\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}}, D(\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}}))$ *is associated with a positive definite selfadjoint operator, the Friedrichs extension of* $H^{\Omega}_{\pi_{\widetilde{\sigma}}}$*, which we also denote by* $H^{\Omega}_{\pi_{\widetilde{\sigma}}}$ *(and its domain by* $D(H^{\Omega}_{\pi_{\widetilde{\sigma}}})$*).*

Clearly, ∇^{Ω} also extends to $D(\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}})$. We denote this extension by ∇^{Ω}.

Corollary 5.2 *Let*

$$\begin{gathered} F(\omega) := g_F(\langle\varphi_1,\omega\rangle, \dots, \langle\varphi_N,\omega\rangle), \qquad \omega \in \Omega^M_X, \\ \varphi_1, \dots, \varphi_N \in D(\mathcal{E}^{X\times M}_{\widetilde{\sigma}}),\ g_F \in C^{\infty}_{\mathrm{b}}(\mathbb{R}^N). \end{gathered} \tag{5.15}$$

Then $F \in D(\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}})$ *and*

$$(\nabla^{\Omega}F)(\omega; x) = \sum_{j=1}^{N} \frac{\partial g_F}{\partial r_j}(\langle\varphi_1,\omega\rangle, \dots, \langle\varphi_N,\omega\rangle)\nabla^{X\times M}\varphi_j(x, s_x).$$

Proof. By approximation this is an immediate consequence of (3.12) and the fact that, for all $1 \le i \le N$,

$$\int \langle|\nabla^{X\times M}\varphi_i|^2_{T(X\times M)}, \omega\rangle\, \pi_{\widetilde{\sigma}}(d\omega) = \mathcal{E}^{X\times M}_{\widetilde{\sigma}}(\varphi_i, \varphi_i). \tag{5.16}$$

Remark 5.2 Let $\mu_{\nu,\widetilde{\sigma}} \in \mathcal{M}_2(\Omega^M_X)$ be given as in (3.18). Then, by Theorem 3.2, (ii)⇒(i), all results above are valid with $\mu_{\nu,\widetilde{\sigma}}$ replacing $\pi_{\widetilde{\sigma}}$. By (5.12) we have

$$H^{\Omega}_{\pi_{\widetilde{\sigma}}} = H^{\Omega}_{\mu_{\nu,\widetilde{\sigma}}} \quad \text{on } \mathcal{F}C^{\infty}_{\mathrm{b}}(\mathfrak{D}_0, \Omega^M_X).$$

We note that the r.h.s. of (5.12) only depends on $\widetilde{\sigma}$ and the Riemannian structure of $X \times M$. The respective Friedrichs extension on $L^2(\mu_{\nu,\widetilde{\sigma}})$ is again denoted by $H^{\Omega}_{\mu_{\nu,\widetilde{\sigma}}}$, however it does necessarily not coincide with $H^{\Omega}_{\pi_{\widetilde{\sigma}}}$.

5.3 The heat semigroup and ergodicity

The results of this subsection are obtained absolutely analogously to the corresponding results of the paper [7], so we omit the proofs.

For $\mu_{\varkappa,\widetilde{\sigma}} \in \mathcal{M}_2(\Omega^M_X)$ let $T^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}(t) := \exp(-tH^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}})$, $t > 0$. Define

$$E(\mathfrak{D}_1, \Omega^M_X) = \mathrm{l.\,h.}\left\{\ \exp(\langle\log(1+\varphi), \cdot\rangle) \mid \varphi \in \mathfrak{D}_1 \right\},$$

where l.h. means the linear hull and

$$\begin{aligned} \mathfrak{D}_1 := \big\{\, \varphi \in D(H^{X\times M}_{\widetilde{\sigma}}) \cap L^1(\widetilde{\sigma}) \mid{} & H^{X\times M}_{\widetilde{\sigma}}\varphi \in L^1(\widetilde{\sigma}) \\ & \text{and } -\delta \le \varphi \le 0 \text{ for some } \delta \in (0,1) \,\big\}. \end{aligned}$$

Proposition 5.1 *Let $\mu_{\varkappa,\widetilde{\sigma}}$ be as in* (3.18). *Assume that $H_{\widetilde{\sigma}}^{X\times M}$ is conservative, i.e.,*

$$\int_{X\times M} (H_{\widetilde{\sigma}}^{X\times M}\varphi)(x,m)\,\widetilde{\sigma}(dx,dm) = 0$$

for all $\varphi \in D(H_{\widetilde{\sigma}}^{X\times M}) \cap L^1(\widetilde{\sigma})$ such that $H_{\widetilde{\sigma}}^{X\times M}\varphi \in L^1(\widetilde{\sigma})$, and suppose that $(H_{\widetilde{\sigma}}^{X\times M}, \mathfrak{D}_0)$ is essentially selfadjoint on $L^2(\widetilde{\sigma})$. Then

$$T^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}(t)\exp(\langle \log(1+\varphi),\cdot\rangle) = \exp(\langle \log(1+e^{-tH_{\widetilde{\sigma}}^{X\times M}}\varphi),\cdot\rangle), \qquad \varphi\in\mathfrak{D}_1, \tag{5.17}$$

$E(\mathfrak{D}_1,\Omega_X^M)\subset D(H^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}})$, and

$$\begin{aligned} &H^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}\exp(\langle \log(1+\varphi),\cdot\rangle)\\ &\quad = \langle (1+\varphi)^{-1}H_{\widetilde{\sigma}}^{X\times M}\varphi,\cdot\rangle \exp(\langle \log(1+\varphi),\cdot\rangle), \qquad \varphi\in\mathfrak{D}_1. \end{aligned}$$

Remark 5.3 (i) The condition of essential selfadjointness of $H_{\widetilde{\sigma}}^{X\times M}$ on $\mathfrak{D}_0$ is fulfilled if X is complete and $|\beta^{\widetilde{\sigma}}|_{T(X\times M)}\in L^p_{\mathrm{loc}}(X\times M; m\otimes\lambda)$ for some $p\geq \dim(X)+1$.

(ii) Since $(\exp(-tH_{\widetilde{\sigma}}^{X\times M}))_{t>0}$ is sub-Markovian (i.e., $0\leq \exp(-tH_{\widetilde{\sigma}}^{X\times M})\varphi\leq 1$ for all $t>0$ and $\varphi\in L^2(\widetilde{\sigma})$, $0\leq\varphi\leq 1$), because $(\mathcal{E}_{\widetilde{\sigma}}^{X\times M}, D(\mathcal{E}_{\widetilde{\sigma}}^{X\times M}))$ is a Dirichlet form, by a simple approximation argument Proposition 5.1 implies that the equality (5.17) holds for $t>0$ and all $\varphi\in L^1(\widetilde{\sigma})$, $-1<\varphi\leq 0$.

Theorem 5.2 *Let the conditions of Proposition* 5.1 *hold. Then $E(\mathfrak{D}_1,\Omega_X^M)$ is an operator core for the Friedrichs extension $H^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}$ on $L^2(\mu_{\varkappa,\widetilde{\sigma}})$. (In other words: $(H^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}, E(\mathfrak{D}_1,\Omega_X^M))$ is essentially selfadjoint on $L^2(\mu_{\varkappa,\widetilde{\sigma}})$.)*

Theorem 5.3 *Suppose that the conditions of Theorem* 3.3 *and Proposition* 5.1 *hold. Then the following assertions are equivalent*:

(i) *$\mu_{\varkappa,\widetilde{\sigma}}=\pi_{z\widetilde{\sigma}}$ for some $z>0$.*

(ii) *$(\mathcal{E}^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}, D(\mathcal{E}^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}))$ is irreducible (i.e., for $F\in D(\mathcal{E}^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}})$, $\mathcal{E}^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}(F,F)=0$ implies that* $F=\mathrm{const}$).

(iii) *$(T^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}(t))_{t>0}$ is irreducible (i.e., if $G\in L^2(\mu_{\varkappa,\widetilde{\sigma}})$ such that $T^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}(t)(GF)=GT^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}(t)F$ for all $F\in L^{\infty}(\mu_{\varkappa,\widetilde{\sigma}})$, $t>0$, then $G=\mathrm{const}$).*

(iv) *If $F\in L^2(\mu_{\varkappa,\widetilde{\sigma}})$ such that $T^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}(t)F=F$ for all $T>0$, then $F=\mathrm{const}$.*

(v) *$T^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}(t)\not\equiv\mathbf{1}$ and ergodic (i.e.,*

$$\int\left(T^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}(t)F-\int F\,d\mu_{\varkappa,\widetilde{\sigma}}\right)^2 d\mu_{\varkappa,\widetilde{\sigma}}\to 0 \quad \text{as t}\to 0$$

for all $F\in L^2(\mu_{\varkappa,\widetilde{\sigma}})$).

(vi) *If $F\in D(H^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}})$ with $H^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}=0$, then $F=\mathrm{const}$.*

Remark 5.4 Let us consider the diffusion process P on $X \times M$ associated to the Dirichlet form $(\mathcal{E}_{\widetilde{\sigma}}^{X\times M}, D(\mathcal{E}_{\widetilde{\sigma}}^{X\times M}))$. This process can be interpreted as distorted Brownian motion on the manifold $X \times M$. More precisely, the diffusion of points $x \in X$ is associated to the Dirichlet form of the measure σ, so that it is distorted Brownian motion on X, and the diffusion of marks s_x, $x \in X$, is associated to the $\widetilde{\nabla}^M$-Dirichlet form of the measure $p(x, dm)$ on M.

The existence of a diffusion process $\mathbf{P}$ corresponding to the Dirichlet form $(\mathcal{E}^\Omega_{\mu_{\varkappa,\widetilde{\sigma}}}, D(\mathcal{E}^\Omega_{\mu_{\varkappa,\widetilde{\sigma}}}))$ follows from [31], and its identification with the independent infinite particle process (on $X \times M$) may be proved by the same arguments as in [7]. By analogy with the case of the process P on $X \times M$, one can call $\mathbf{P}$ distorted Brownian motion on Ω_X^M.

6 Intrinsic Dirichlet operator and second quantization

In this section, we want to describe the Fock space realization of the marked Poisson spaces and show that $H^\Omega_{\pi_{\widetilde{\sigma}}}$ is the second quantization of the operator $H_{\widetilde{\sigma}}^{X\times M}$.

6.1 Marked Poisson gradient and chaos decomposition

Let us define another "gradient" on functions $F\colon \Omega_X^M \to \mathbb{R}$, which has specific useful properties on the marked Poisson space.

Definition 6.1 For any $F \in \mathcal{F}C_{\mathrm{p}}^\infty(\mathfrak{D}_0, \Omega_X^M)$ we define the marked Poisson gradient ∇^{MP} as

$$(\nabla^{\mathrm{MP}} F)(\omega, (x,m)) := F(\omega + \varepsilon_{(x,m)}) - F(\omega), \qquad \omega \in \Omega_X^M,\ (x,m) \in X \times M.$$

Let us mention that the operation

$$\Omega_X^M \ni \omega \mapsto \omega + \varepsilon_{(x,m)} \in \Omega_X^M$$

is a $\pi_{\widetilde{\sigma}}$-a.e. well-defined map because of the property

$$\pi_{\widetilde{\sigma}}\big(\{\omega = (\gamma, s) \in \Omega_X^M \mid x \in \gamma\}\big) = 0$$

for an arbitrary $x \in X$ (which easily follows from the construction of $\pi_{\widetilde{\sigma}}$). We consider ∇^{MP} as a mapping

$$\nabla^{\mathrm{MP}}\colon \mathcal{F}C_{\mathrm{p}}^\infty(\mathfrak{D}_0, \Omega_X^M) \ni F \mapsto \nabla^{\mathrm{MP}} F \in L^2(\widetilde{\sigma}) \otimes L^2(\pi_{\widetilde{\sigma}})$$

that corresponds to using the Hilbert space $L^2(\widetilde{\sigma})$ as a tangent space at any point $\omega \in \Omega_X^M$. Thus, for any $\varphi \in \mathfrak{D}_0$, we can introduce the directional derivative

$$\begin{aligned}(\nabla^{\mathrm{MP}}_\varphi F)(\omega) &= \langle \nabla^{\mathrm{MP}} F(\omega), \varphi \rangle_{L^2(\widetilde{\sigma})} \\ &= \int_{X\times M} (F(\omega + \varepsilon_{(x,m)}) - F(\omega))\varphi(x,m)\, \widetilde{\sigma}(dx, dm).\end{aligned}$$

The most important feature of the marked Poisson gradient is that it produces (via a corresponding "integration by parts formula") the orthogonal system of Charlier polynomials on $(\Omega_X^M, \mathcal{B}(\Omega_X^M), \pi_{\widetilde{\sigma}})$. Below, we describe this construction in detail using the isomorphism between $L^2(\pi_{\widetilde{\sigma}})$ and the symmetric Fock space (see [21, 25, 30])

Let $\mathcal{F}(L^2(\widetilde{\sigma}))$ denote the symmetric Fock space over $L^2(\widetilde{\sigma})$:

$$\mathcal{F}(L^2(\widetilde{\sigma})) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(L^2(\widetilde{\sigma}))n!,$$

where

$$\mathcal{F}_n(L^2(\widetilde{\sigma})) := (L^2(\widetilde{\sigma}))^{\widehat{\otimes} n} = \hat{L}^2((X \times M)^n, \widetilde{\sigma}^{\otimes n}), \qquad n \in \mathbb{N},$$
$$\mathcal{F}_0(L^2(\widetilde{\sigma})) := \mathbb{R},$$

$\widehat{\otimes}$ denoting the symmetric tensor product. Thus, for each $F = (f^{(n)})_{n=0}^{\infty} \in \mathcal{F}(L^2(\widetilde{\sigma}))$

$$\|F\|^2_{\mathcal{F}(L^2(\widetilde{\sigma}))} = \sum_{n=0}^{\infty} |f^{(n)}|_{\hat{L}^2(\widetilde{\sigma}^{\otimes n})} n!.$$

By $\mathcal{F}_{\mathrm{fin}}(\mathfrak{D}_0)$ we denote the dense subset of $\mathcal{F}(L^2(\widetilde{\sigma}))$ consisting of finite sequences $(f^{(n)})_{n=0}^{N}$, $n \in \mathbb{Z}_+$, such that each $f^{(n)}$ belongs to $\mathcal{F}_n(\mathfrak{D}_0) := \mathrm{a}.\mathfrak{D}_0^{\widehat{\otimes} n}$, the n-th symmetric algebraic tensor power of $\mathfrak{D}_0$:

$$\mathrm{a}.\mathfrak{D}_0^{\widehat{\otimes} n} := \mathrm{l.\,h.}\{\varphi_1 \widehat{\otimes} \cdots \widehat{\otimes} \varphi_n \mid \varphi_i \in \mathfrak{D}_0\}.$$

In virtue of the polarization identity, the latter set is spanned just by the vectors of the form $\varphi^{\otimes n}$ with $\varphi \in \mathfrak{D}_0$.

Now, we define a linear mapping

$$\mathcal{F}_{\mathrm{fin}}(\mathfrak{D}_0) \ni F = (f^{(n)})_{n=0}^{N} \mapsto IF = (IF)(\omega) = \sum_{n=0}^{N} Q_n(f^{(n)}; \omega) \in \mathcal{FP}(\mathfrak{D}_0, \Omega_X^M) \tag{6.1}$$

by using the following recursion relation:

$$\begin{aligned} Q_{n+1}(\varphi^{\otimes(n+1)}; \omega) = {} & Q_n(\varphi^{\otimes n}; \omega)(\langle \omega, \varphi \rangle - \langle \varphi \rangle_{\widetilde{\sigma}}) \\ & - nQ_n(\varphi^{\otimes(n-1)} \widehat{\otimes} (\varphi^2), \omega) - nQ_{n-1}(\varphi^{\otimes(n-1)}; \omega) \langle \varphi^2 \rangle_{\widetilde{\sigma}}, \\ & Q_0(1, \omega) = 1, \quad \varphi \in \mathfrak{D}_0. \end{aligned} \tag{6.2}$$

Here, we have set $\langle \varphi \rangle_{\widetilde{\sigma}} := \int \varphi \, d\widetilde{\sigma}$. Notice that, since $\mathfrak{D}_0$ is an algebra under pointwise multiplication of functions, the latter definition is correct.

It is not hard to see that the mapping (6.1) is one-to-one. Moreover, the following proposition holds:

Proposition 6.1 *The mapping* (6.1) *can be extended by continuity to a unitary isomorphism between the spaces* $\mathcal{F}(L^2(\widetilde{\sigma}))$ *and* $L^2(\pi_{\widetilde{\sigma}})$.

For each $\varphi \in \mathfrak{D}_0$, let us define the creation and annihilation operators in $\mathcal{F}(L^2(\widetilde{\sigma}))$ by

$$a^+(\varphi)\psi^{\otimes n} = \varphi\widehat{\otimes}\psi^{\otimes n}, \qquad a^-(\varphi)\psi^{\otimes n} = n(\varphi,\psi)_{L^2(\widetilde{\sigma})}\psi^{\otimes(n-1)}, \qquad \psi \in \mathfrak{D}_0.$$

We will denote by the same letters the images of these operators under the unitary I.

Proposition 6.2 *We have, for each* $\varphi \in \mathfrak{D}_0$,

$$a^-(\varphi) = \nabla^{\mathrm{MP}}_\varphi, \qquad a^+(\varphi) = \nabla^{\mathrm{MP}*}_\varphi.$$

In particular,

$$Q_n(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_n;\omega) = (\nabla^{\mathrm{MP}*}_{\varphi_1}\cdots\nabla^{\mathrm{MP}*}_{\varphi_n}\mathbf{1})(\omega), \qquad \omega \in \Omega_X^M.$$

Finally, for each $\varphi \in \mathfrak{D}_0$ we introduce the Poisson exponential

$$e(\varphi;\cdot) := \sum_{n=0}^\infty \frac{1}{n!}\, Q_n(\varphi^{\otimes n};\cdot) = I(\operatorname{Exp}\varphi),$$

where

$$\operatorname{Exp}\varphi = \left(\frac{1}{n!}\,\varphi^{\otimes n}\right)_{n=0}^\infty.$$

Then, one can show that, for $\varphi > -1$,

$$e(\varphi;\omega) = \exp\left[\langle\log(1+\varphi),\omega\rangle - \langle\varphi\rangle_{\widetilde{\sigma}}\right], \qquad \omega \in \Omega_X^M. \tag{6.3}$$

6.2 Second quantization on the marked Poisson space

Let B be a contraction on $L^2(\widetilde{\sigma})$, i.e., $B \in \mathcal{L}(L^2(\widetilde{\sigma}), L^2(\widetilde{\sigma}))$, $\|B\| \le 1$. Then, we can define the operator $\operatorname{Exp} B$ as the contraction on $\mathcal{F}(L^2(\widetilde{\sigma}))$ given by

$$\begin{aligned}&\operatorname{Exp} B \restriction \mathcal{F}_n(L^2(\widetilde{\sigma})) := B \otimes \cdots \otimes B \quad (n \text{ times}),\ n \in \mathbb{N},\\ &\operatorname{Exp} B \restriction \mathcal{F}_0(L^2(\widetilde{\sigma})) := \mathbf{1}.\end{aligned}$$

For any selfadjoint positive operator A in $L^2(\widetilde{\sigma})$, we have a contraction semigroup e^{-tA}, $t \ge 0$, and it is possible to introduce a positive selfadjoint operator $d\operatorname{Exp} A$ as the generator of the semigroup $\operatorname{Exp}(e^{-tA})$, $t \ge 0$:

$$\operatorname{Exp}(e^{-tA}) = \exp(-td\operatorname{Exp} A). \tag{6.4}$$

The operator $d\operatorname{Exp} A$ is called the second quantization of A. We denote by H_A^{MP} the image of the operator $d\operatorname{Exp} A$ in the marked Poisson space $L^2(\pi_{\widetilde{\sigma}})$.

Theorem 6.1 *Let* $\mathfrak{D}_0 \subset \operatorname{Dom} A$. *Then, the symmetric bilinear form corresponding to the operator* H_A^{MP} *has the following representation:*

$$(H_A^{\mathrm{MP}}F_1, F_2)_{L^2(\pi_{\widetilde{\sigma}})} = \int_{\Omega_X^M} (\nabla^{\mathrm{MP}}F_1, A\nabla^{\mathrm{MP}}F_2)_{L^2(\widetilde{\sigma})}\,\pi_{\widetilde{\sigma}}(d\omega) \tag{6.5}$$

for all $F_1, F_2 \in \mathcal{FP}(\mathfrak{D}_0, \Omega_X^M)$.

Remark 6.1 The bilinear form (6.5) uses the marked Poisson gradient ∇^{MP} and a coefficient operator $A > 0$. We will call

$$\mathcal{E}^{\mathrm{MP}}_{\pi_{\widetilde{\sigma}},A}(F_1, F_2) = \int_{\Omega_X^M} (\nabla^{\mathrm{MP}} F, A\nabla^{\mathrm{MP}} G)_{L^2(\widetilde{\sigma})}\, \pi_{\widetilde{\sigma}}(d\omega)$$

the marked Poisson pre-Dirichlet form with coefficient A.

Proof of Theorem 5.1. The proof is analogous to that of Theorem 5.1 in [7]. Using again the fact that $\mathfrak{D}_0$ is an algebra under pointwise multiplication, one easily concludes that, for any $F \in \mathcal{FP}(\mathfrak{D}_0, \Omega_X^M)$ and any $\omega \in \Omega_X^M$, the gradient $\nabla^{\mathrm{MP}} F(\omega, (x, m))$ is a function in $\mathfrak{D}_0$ and hence

$$(\nabla^{\mathrm{MP}} F, A\nabla^{\mathrm{MP}} G)_{L^2(\widetilde{\sigma})} \in \mathcal{FP}(\mathfrak{D}_0, \Omega_X^M),$$

so that the form (6.5) is well-defined. Then, one verifies the formula (6.5) by using Propositions 5.1, 5.2 and the explicit formula for $d\operatorname{Exp} A$ on $\mathcal{F}_n(\mathfrak{D}_0)$:

$$d\operatorname{Exp} A\, \varphi^{\otimes n} = n(A\varphi)\widehat{\otimes}\varphi^{\otimes(n-1)}, \qquad \varphi \in \mathfrak{D}_0. \quad \blacksquare$$

6.3 The intrinsic Dirichlet operator as a second quantization

The following two theorems are again analogous to the corresponding results (Theorems 5.2 and 5.3) in [7], so we omit their proofs.

Let us consider the special case of the second quantization operator $d\operatorname{Exp} A$ where the operator A coincides with the Dirichlet operator $H^{X\times M}_{\widetilde{\sigma}}$.

Theorem 6.2 *We have the equality*

$$H^{\mathrm{MP}}_{H^{X\times M}_{\widetilde{\sigma}}} = H^{\Omega}_{\pi_{\widetilde{\sigma}}}$$

on the dense domain $\mathcal{F}C^\infty_{\mathrm{p}}(\mathfrak{D}_0, \Omega_X^M)$. In particular, for all $F_1, F_2 \in \mathcal{F}C^\infty_{\mathrm{p}}(\mathfrak{D}_0, \Omega_X^M)$

$$\int_{\Omega_X^M} \langle \nabla^\Omega F_1(\omega), \nabla^\Omega F_2(\omega)\rangle_{T_\omega(\Omega_X^M)}\, \pi_{\widetilde{\sigma}}(d\omega)$$
$$= \int_{\Omega_X^M} (\nabla^{\mathrm{MP}} F_1(\omega), H^{X\times M}_{\widetilde{\sigma}} \nabla^{\mathrm{MP}} F_2(\omega))_{L^2(\widetilde{\sigma})}\, \pi_{\widetilde{\sigma}}(d\omega),$$

or

$$\nabla^{\Omega *}\nabla^\Omega = \nabla^{\mathrm{MP}*} H^{X\times M}_{\widetilde{\sigma}} \nabla^{\mathrm{MP}}$$

as an equality on $\mathcal{F}C^\infty_{\mathrm{p}}(\mathfrak{D}_0, \Omega_X^M)$.

Theorem 6.3 *Suppose that the operator $H^{X\times M}_{\widetilde{\sigma}}$ is essentially selfadjoint on the domain $\mathfrak{D}_0 \subset \operatorname{Dom}(H^{X\times M}_{\widetilde{\sigma}})$. Then, the intrinsic Dirichlet operator $H^\Omega_{\pi_{\widetilde{\sigma}}}$ is essentially selfadjoint on the domain $\mathcal{F}C^\infty_{\mathrm{b}}(\mathfrak{D}_0, \Omega_X^M)$.*

Remark 6.2 Notice that in Theorem 6.3 we do not suppose the operator $H_{\widetilde\sigma}^{X\times M}$ to be conservative. So, this theorem is a generalization of Theorem 5.2 in the special case where $\mu_{\varkappa,\widetilde\sigma}=\pi_{\widetilde\sigma}$.

Corollary 6.1 *Suppose that the condition of Theorem* 6.3 *is satisfied and let* $T_{\pi_{\widetilde\sigma}}^{\Omega}(t)=\exp(-tH_{\pi_{\widetilde\sigma}}^{\Omega})$, $t>0$. *Then, for each* $\varphi\in\mathfrak{D}_0$, $\varphi>-1$, *we have*

$$T_{\pi_{\widetilde\sigma}}^{\Omega}(t)\exp(\langle\log(1+\varphi),\cdot\rangle)=\exp\left[\langle\log(1+e^{-tH_{\widetilde\sigma}^{X\times M}}\varphi),\cdot\rangle-\langle(e^{-tH_{\widetilde\sigma}^{X\times M}}-\mathbf{1})\varphi\rangle_{\widetilde\sigma}\right]. \quad (6.6)$$

Proof. The formula (6.6) follows from Proposition 6.1, (6.3), (6.4) and Theorems 6.2 and 6.3. ■

Remark 6.3 If $H_{\widetilde\sigma}^{X\times M}$ is conservative, then

$$\int(e^{-tH_{\widetilde\sigma}^{X\times M}}-\mathbf{1})\varphi\, d\widetilde\sigma=0 \qquad \text{for all } t\ge 0,$$

and so in this case (6.6) coincides with (5.17) for $\varphi\in\mathfrak{D}_0$, $\varphi>-1$.

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